MATH 141 EXAM #1 KEY (FALL 2020)

1 By the Inverse Function Theorem $(f^{-1})'(f(a)) = 1/f'(a)$, so find a such that $f(a) = 3$:

$$f(a) = 3 \;\Rightarrow\; a^3 + 4a + 4 = 9 \;\Rightarrow\; a = 1.$$

Also

$$f'(x) = \frac{3x^2 + 4}{2\sqrt{x^3 + 4x + 4}} \;\Rightarrow\; f'(1) = \frac{7}{6}.$$

Then

$$(f^{-1})'(3) = (f^{-1})'(f(1)) = \frac{1}{f'(1)} = \frac{6}{7}.$$

2a $\dfrac{d}{dx}\left[x - \ln(2 + e^x)\right] = 1 - \dfrac{e^x}{2 + e^x}$

2b $\dfrac{d}{dt} e^{2t\ln(1-3t)} = (1 - 3t)^{2t}\left[2\ln(1 - 3t) - \dfrac{6t}{1 - 3t}\right]$

2c $\dfrac{d}{dr}(-r\log_6 r) = -\dfrac{d}{dr}\left(\dfrac{r\ln r}{\ln 6}\right) = -\dfrac{1 + \ln r}{\ln 6}$

2d $\dfrac{2y}{1 + (y^2)^2} + \dfrac{1}{\sqrt{1 - (\sqrt{y})^2}} \cdot \dfrac{1}{2\sqrt{y}} = \dfrac{2y}{1 + y^4} + \dfrac{1}{2\sqrt{y - y^2}}$

2e $6e^{2x}\tanh^2(e^{2x})\,\mathrm{sech}^2(e^{2x})$

3 For $y(x) = 2e^x - 1$ we have $y'(x) = 2e^x$, so $y'(\ln 3) = 6$ is the slope of the line. The line contains point $(\ln 3, y(\ln 3)) = (\ln 3, 5)$, so equation is

$$y = 6x - 6\ln 3 + 5.$$

4a With a long division we have

$$\int_0^3 \frac{2p - 1}{p + 1}\,dp = \int_0^3 \left(2 - \frac{3}{p + 1}\right) dp = \left[2p - 3\ln|p + 1|\right]_0^3 = 6 - 3\ln 4.$$

4b Use a given formula:

$$\frac{1}{2}\int \frac{dy}{y\sqrt{y^2 - 5^2}} = \frac{1}{2} \cdot \frac{1}{5}\sec^{-1}\left|\frac{y}{5}\right| + C = \frac{1}{10}\sec^{-1}\left|\frac{y}{5}\right| + C.$$

4c Generally $\int e^{ax}dx = e^{ax}/a$, so,

$$\int 4^{-2x}dx = \int e^{-2x\ln 4}dx = \frac{1}{-2\ln 4}e^{-2x\ln 4} + C = -\frac{4^{-2x}}{2\ln 4} + C.$$

4d Let $u = 3 + 2e^x$, so $e^x\,dx = \frac{1}{2}du$ and the integral becomes

$$\int_5^{11} \frac{1/2}{u}\,du = \frac{1}{2}\big[\ln|u|\big]_5^{11} = \frac{1}{2}\ln\left(\frac{11}{5}\right).$$

5 $f'(x) = 0$ implies $e^{-x^2}(1 - 2x^2) = 0$ implies $x = \pm\frac{1}{\sqrt{2}}$, the critical points for f. Since $f'(-1) < 0$, $f'(0) > 0$, and $f'(1) < 0$, the Intermediate Value Theorem implies $f' < 0$ on $(-\infty, -\frac{1}{\sqrt{2}})$, $f' > 0$ on $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, and $f' < 0$ on $(\frac{1}{\sqrt{2}}, \infty)$. By the First Derivative Test f has a minimum at $x = -\frac{1}{\sqrt{2}}$ and a maximum at $x = \frac{1}{\sqrt{2}}$.

6a With L'Hôpital's Rule,

$$\lim_{x\to 0^+} (e^{3x} + x)^{1/x} = \exp\left[\lim_{x\to 0^+} \frac{\ln(e^{3x} + x)}{x}\right] \overset{\text{LR}}{=} \lim_{x\to 0^+}\left[\frac{3e^{3x} + 1}{e^{3x} + x}\right] = e^4.$$

6b Use L'Hôpital's Rule twice:

$$\begin{aligned}
\lim_{t\to 1^+} \exp\left[\frac{\sin \pi t}{2} \cdot \ln(t-1)\right] &= \exp\left[\lim_{t\to 1^+} \frac{\ln(t-1)}{2\csc \pi t}\right] \\
&\overset{\text{LR}}{=} \exp\left[\lim_{t\to 1^+} \frac{\sin \pi t \tan \pi t}{2\pi(1-t)}\right] \\
&\overset{\text{LR}}{=} \exp\left[\lim_{t\to 1^+} \frac{\cos \pi t \tan \pi t + \sin \pi t \sec^2 \pi t}{-2}\right] \\
&= e^0 = 1.
\end{aligned}$$