Math 141 Exam #4 Key (Fall 2019)

1 Let $f(x) = \sqrt{x}$, and find the 3rd-order Taylor polynomial with center 9:

$$\begin{aligned} p_3(x) &= f(9) + f'(9)(x-9) + \frac{f''(9)}{2!}(x-9)^2 + \frac{f'''(9)}{3!}(x-9)^3 \\ &= 3 + \frac{x-9}{6} - \frac{(x-9)^2}{216} + \frac{(x-9)^3}{3888}. \end{aligned}$$

Now,

$$\sqrt{9.3} = f(9.3) \approx p_3(9.3) = 3 + \frac{0.3}{6} - \frac{0.3^2}{216} + \frac{0.3^3}{3888} \approx 3.049589.$$

2a Apply Ratio Test:

$$\lim_{n\to\infty} \left| \frac{2^{2(n+1)}x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{2^{2n}x^n} \right| = |x| \lim_{n\to\infty} \frac{4n^2}{(n+1)^2} = 4|x|.$$

Series converges if $|x| < 1/4$, so interval of convergence contains $\left(-\frac{1}{4}, \frac{1}{4}\right)$. Check endpoints.

At $x = 1/4$: series becomes $\sum 1/n^2$, a convergent p-series. At $x = -1/4$: series becomes $\sum(-1)^n/n^2$, which converges by the Alternating Series Test.

Interval of convergence is $\left[-\frac{1}{4}, \frac{1}{4}\right]$.

2b Ratio Test:

$$\lim_{n\to\infty} \left| \frac{(x+1)^{n+1}}{(n+1)\cdot 6^{n+1}} \cdot \frac{n \cdot 6^n}{(x+1)^n} \right| = |x+1| \lim_{n\to\infty} \frac{n}{6n+6} = \frac{|x+1|}{6}.$$

Series converges if $|x+1| < 6$, so interval of convergence contains $(-7, 5)$. Check endpoints.

At $x = 5$ series becomes $\sum 1/n$, which diverges. At $x = -7$ series becomes $\sum(-1)^n/n$, which converges by the Alternating Series Test.

Interval of convergence is $\left[-7, 5\right)$.

2c Ratio Test:

$$\lim \left| \frac{(n+1)!(x-3)^{n+1}}{n!(x-3)^n} \right| = \lim_{n\to\infty} (n+1)|x-3| = \begin{cases} \infty, & x \neq 3 \\ 0, & x = 3. \end{cases}$$

The series only converges at $\{3\}$.

3 Using the formula for a convergent geometric series,

$$f(x) = 2x \cdot \frac{1}{1-x^4} = 2x \sum_{n=0}^{\infty} x^{4n} = \sum_{n=0}^{\infty} 2x^{4n+1}.$$

Apply the Ratio Test:

$$\lim_{n\to\infty} \left| \frac{2x^{4(n+1)+1}}{2x^{4n+1}} \right| = 2x^4.$$

Series converges if $2x^4 < 1$, so $\left(-1/\sqrt[4]{2}, 1/\sqrt[4]{2}\right)$ is contained in the interval of convergence. Since the series diverges at the endpoints, $\left(-2^{-1/4}, 2^{-1/4}\right)$ is the interval of convergence.

4 Binomial series:

$$(1+x)^{1/2} = \sum_{n=0}^{\infty} \binom{1/2}{n} x^n = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots,$$

with interval of convergence $(-1, 1)$.

5 We have

$$\lim_{x\to 0} \frac{1 - \left(1 - \dfrac{x^2}{2} + \dfrac{x^4}{24} - \cdots\right)}{1 + x - \left(1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + \cdots\right)} = \lim_{x\to 0} \frac{\dfrac{x^2}{2} - \dfrac{x^4}{24} + \cdots}{-\dfrac{x^2}{2} - \dfrac{x^3}{6} - \cdots} = \lim_{x\to 0} \frac{\dfrac{1}{2} - \dfrac{x^2}{24} + \cdots}{-\dfrac{1}{2} - \dfrac{x}{6} - \cdots} = -1.$$

6 From the table provided we have $e^x = \sum_{n=0}^{\infty} x^n/n!$ for all $x \in (-\infty, \infty)$, and so

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$$

for all x. Now,

$$\int \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n} \right) dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} x^{2n+1} + c$$

for all x and arbitrary constant c. Thus, by the Fundamental Theorem of Calculus,

$$\int_0^{1/3} e^{-x^2}\,dx = \int_0^{1/3} \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n} \right) dx = \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} x^{2n+1} \right]_0^{1/3} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} \left(\frac{1}{3}\right)^{2n+1}.$$

We have arrived at an alternating series $\sum (-1)^n b_n$ with

$$b_n = \frac{1}{n!(2n+1)} \left(\frac{1}{3}\right)^{2n+1}$$

for $n \geq 0$. The first few b_n values are

$$b_0 = \tfrac{1}{3}, \quad b_1 = \tfrac{1}{81}, \quad b_2 = \tfrac{1}{2430} > 10^{-4}, \quad b_3 = \tfrac{1}{91{,}854} < 10^{-4},$$

so by the Alternating Series Estimation Theorem the approximation

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} \left(\frac{1}{3}\right)^{2n+1} \approx b_0 - b_1 + b_2 \approx 0.321399$$

will have an absolute error that is less than $b_3 < 10^{-4}$. Hence the approximation

$$\int_0^{1/3} e^{-x^2}\,dx \approx \frac{1}{3} - \frac{1}{81} + \frac{1}{2430}$$

has an absolute error less than 10^{-4}.

7a Setting $x = f(t) = \sqrt[3]{t^8 - 8}$ and $y = g(t) = \sqrt{t^4 + 1}$, so

$$f'(t) = \frac{1}{3}(t^8 - 8)^{-2/3}(8t^7) \quad \text{and} \quad g'(t) = \frac{1}{2}(t^4 + 1)^{-1/2}(4t^3).$$

Slope is given by $g'(0)/f'(0)$, which here is undefined since $f'(0) = 0$.

7b Since $y = \sqrt{t^4 + 1}$ implies $t^4 = y^2 - 1$, we obtain

$$x = \sqrt[3]{t^8 - 8} = \sqrt[3]{(y^2 - 1)^2 - 8} = \sqrt[3]{y^4 - 2y^2 - 7}.$$

8 The set-up is thus:

$$(x, y) = \left(1 - \tfrac{1}{10}t\right)(-2, 5) + \tfrac{1}{10}t\,(2, -1)$$

for $0 \leq t \leq 10$. Equivalently we may write

$$(x, y) = \left(\tfrac{2}{5}t - 2, -\tfrac{3}{5}t + 5\right), \quad t \in [0, 10].$$

9 As given here, r can never be 0. Thus we may safely divide by r:

$$\frac{2}{4r\cos\theta + 3r\sin\theta} = 1 \;\Rightarrow\; \frac{2}{4x + 3y} = 1 \;\Rightarrow\; 4x + 3y = 2.$$

This is a line, of course.