Math 141 Exam #3 Key (Fall 2019)

1a Use the continuity of the logarithm and L'Hôpital's rule:

$$\lim_{n\to\infty} \ln\left(n\sin\frac{1}{n}\right) = \ln\left(\lim_{n\to\infty} n\sin\frac{1}{n}\right) \overset{\text{LR}}{=} \ln\left(\lim_{n\to\infty} \frac{-\cos(1/n)/n^2}{-1/n^2}\right)$$
$$= \ln\left(\lim_{n\to\infty} \cos\frac{1}{n}\right) = \ln(\cos 0) = \ln 1 = 0.$$

1b The sequence converges:

$$\lim_{n\to\infty} (e^{3n+4})^{1/n} = \lim_{n\to\infty} e^{3+4/n} = e^3.$$

2a Sequence $(a_n)_{n=0}^{\infty}$ is increasing if and only if $a_{n+1} > a_n$ for all $n \geq 0$, and since

$$a_{n+1} > a_n \quad\Leftrightarrow\quad \frac{1}{3}a_n + 6 > a_n \quad\Leftrightarrow\quad a_n < 9,$$

we can confirm (a_n) is increasing if we can show $a_n < 9$ is true for all $n \geq 0$

Clearly $0 < a_0 < 9$. Now, for arbitrary $n \geq 0$ suppose that $0 < a_n < 9$. Then

$$a_{n+1} = \frac{1}{3}a_n + 6 < \frac{1}{3}\cdot 9 + 6 = 9 \quad\text{and}\quad a_{n+1} = \frac{1}{3}a_n + 6 > \frac{1}{3}\cdot 0 + 6 > 0$$

and we conclude by induction that $0 < a_n < 9$ for all $n \geq 0$. Thus (a_n) is bounded, and also increasing.

2b Because (a_n) is an increasing bounded sequence, the Monotone Convergence Theorem implies that the sequence converges. That is, the limit $\lim_{n\to\infty} a_n = \alpha$ for some $\alpha \in \mathbb{R}$, and with the given recurrence relation we obtain

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\left(\frac{1}{3}a_n + 6\right) \quad\Rightarrow\quad \alpha = \frac{1}{3}\alpha + 6 \quad\Rightarrow\quad \alpha = 9.$$

The limit of the sequence is 9.

3 Reindex to obtain

$$\sum_{n=1}^{\infty} \frac{8}{4^n} = \sum_{n=0}^{\infty} \frac{8}{4^{n+1}} = 2\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n = 2\cdot\frac{1}{1-1/4} = \frac{8}{3}.$$

4 Partial fraction decomposition gives

$$\frac{20}{25n^2+15n-4} = \frac{4}{5n-1} - \frac{4}{5n+4}.$$

The nth partial sum is

$$s_n = \sum_{k=1}^{n} \left(\frac{4}{5k-1} - \frac{4}{5k+4} \right) = 1 - \frac{4}{5n+4},$$

and so

$$\sum_{n=1}^{\infty} \frac{20}{25n^2+15n-4} = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{4}{5n+4} \right) = 1.$$

The series converges.

5 We have

$$5.1\overline{32} = 5.1 + \frac{32}{10^3} + \frac{32}{10^5} + \frac{32}{10^7} + \cdots = 5.1 + \sum_{n=0}^{\infty} \frac{32}{10^{2n+3}}$$
$$= 5.1 + \sum_{n=0}^{\infty} \frac{32}{1000} \left(\frac{1}{100} \right)^n = 5.1 + \frac{32/1000}{1-1/100} = \frac{5081}{990}.$$

6a Compare to $\sum (4/5)^n$. Since

$$\lim_{n\to\infty} \frac{\dfrac{4^n}{5^n-6}}{\dfrac{4^n}{5^n}} = \lim_{n\to\infty} \frac{5^n}{5^n-6} = \lim_{n\to\infty} \frac{1}{1-6/5^n} = \frac{1}{1-0} = 1 \in (0,\infty),$$

the Limit Comparison Test indicates that $\sum(4/5)^n$ and the given series either both converge or both diverge. However, since $\sum(4/5)^n$ is a convergent geometric series, we conclude that the given series also converges.

6b For $n \geq 1$ we have $2^{\ln n} \leq e^{\ln n} = n$, so $1/2^{\ln n} > \frac{1}{n}$, and since $\sum 1/n$ diverges by the p-Series Test, the Comparison Test implies that the given series diverges also.

6c Since

$$\lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} \sqrt[k]{\left(\frac{k^2}{2k^2+1} \right)^k} = \lim_{k\to\infty} \frac{k^2}{2k^2+1} = \frac{1}{2} < 1,$$

the Root Test implies that the given series converges.

6d Since

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)! - n!}{(n+2)! - (n+1)!} \right| = \lim_{n\to\infty} \frac{n}{(n+1)^2} = 0 < 1,$$

the Ratio Test implies that the given series converges.

6e Apply the Integral Test: with the substitution $u = \sqrt{x}$ we have

$$\int_1^\infty \frac{1}{\sqrt{x}e^{\sqrt{x}}}dx = \int_1^\infty \frac{2}{e^u}\,du = \lim_{t\to\infty}\left[-\frac{2}{e^u}\right]_1^t = \lim_{t\to\infty}\left(\frac{2}{e} - \frac{2}{e^t}\right) = \frac{2}{e},$$

and since the integral converges, we conclude that the given series also converges.

6f Since

$$\begin{aligned}\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left(\frac{1\cdot 3\cdot 5\cdots[2(n+1)-1]}{[2(n+1)-1]!}\cdot\frac{(2n-1)!}{1\cdot 3\cdot 5\cdots(2n-1)}\right)\\ &= \lim_{n\to\infty}\frac{2(n+1)-1}{2n(2n+1)} = \lim_{n\to\infty}\frac{2n+1}{4n^2+2n} = 0,\end{aligned}$$

the series converges by the Ratio Test.

7a Since $(1/n^{2/3})$ is a decreasing sequence of nonnegative values such that $1/n^{2/3} \to 0$ as $n \to \infty$, the series converges by the Alternating Series Test. Since $\sum 1/n^{2/3}$ diverges by the p-Series Test, the given series is conditionally convergent.

7b For all $k \geq 1$ we have

$$0 < b_k = \frac{1}{2\sqrt{k}-1},$$

with $b_{k+1} < b_k$ and $b_k \to 0$ as $k \to \infty$. The Alternating Series Test thus implies the series given converges. However,

$$b_k = \frac{1}{2\sqrt{k}-1} > \frac{1}{2\sqrt{k}}$$

for all $k \geq 1$, and since $\sum \frac{1}{2\sqrt{k}}$ diverges by the p-Series Test, we conclude that

$$\sum_{k=1}^{\infty}\frac{1}{2\sqrt{k}-1}$$

diverges by the Comparison Test. Therefore the given series is conditionally convergent.