MATH 141 EXAM #2 KEY (FALL 2019)

1a Let I be the integral. Apply integration by parts twice, starting with $u = x^2$, $v' = \cos 4x$, to get

$$\begin{aligned} I &= \frac{x^2}{4}\sin 4x - \int \frac{1}{2}x\sin 4x\,dx = \frac{x^2}{4}\sin 4x - \left(-\frac{x}{8}\cos 4x + \frac{1}{8}\int \cos 4x\,dx\right) \\ &= \frac{x^2}{4}\sin 4x + \frac{x}{8}\cos 4x - \frac{1}{32}\sin 4x + C. \end{aligned}$$

1b Let I be the integral. Integrate by parts twice, with $u = e^{-x}$, $v' = \sin 3x$ the first time, and $u = e^{-x}$, $v' = \cos 3x$ the second time, we get

$$\begin{aligned} I &= -\frac{1}{3}e^{-x}\cos 3x - \frac{1}{3}\int e^{-x}\cos 3x\,dx \\ &= -\frac{1}{3}e^{-x}\cos 3x - \frac{1}{3}\left(\frac{1}{3}e^{-x}\sin 3x + \frac{1}{3}\int e^{-x}\sin 3x\,dx\right) \\ &= -\frac{1}{3}e^{-x}\cos 3x - \frac{1}{9}e^{-x}\sin 3x - \frac{1}{9}I. \end{aligned}$$

Solving for I gives

$$I = -\frac{1}{10}e^{-x}\big(3\cos 3x + \sin 3x\big) + C.$$

2 By the Fundamental Theorem of Calculus

$$f'(x) = \frac{d}{dx}\int_e^x \sqrt{\ln^2 t - 1}\,dt = \sqrt{\ln^2 x - 1}.$$

Arc length L is

$$\begin{aligned} L &= \int_e^{e^4} \sqrt{1 + [f'(x)]^2}\,dx = \int_e^{e^4} \sqrt{1 + \left(\sqrt{\ln^2 x - 1}\right)^2}\,dx \\ &= \int_e^{e^4} \ln x\,dx = \big[x\ln x - x\big]_e^{e^4} = 3e^4, \end{aligned}$$

applying integration by parts to the last integral.

3a Let $u = \cos 6t$, so

$$\begin{aligned} \int \sin^5 6t\cos^2 6t\,dt &= \int (1 - \cos^2 6t)^2\cos^2 6t\sin 6t\,dt \\ &= -\frac{1}{6}(u^6 - 2u^4 + u^2)\,du \\ &= -\frac{1}{6}\left(\frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3\right) + C \\ &= -\frac{1}{42}\cos^7 6t + \frac{1}{15}\cos^5 6t - \frac{1}{18}\cos^3 6t + C. \end{aligned}$$

3b Let $u = \cot 4r$ to get

$$\int_1^0 -\frac{1}{4}u^4\,du = \frac{1}{4}\left[\frac{1}{5}u^5\right]_0^1 = \frac{1}{20}.$$

4a Let $x = 2\sin\theta$ to get

$$\int_0^{\pi/4} \frac{4\sin^2\theta}{\sqrt{4\cos^2\theta}} \cdot 2\cos\theta\,d\theta = \int_0^{\pi/4} 4\sin^2\theta\,d\theta = 4\left(\left[-\frac{\sin\theta\cos\theta}{2}\right]_0^{\pi/4} + \frac{1}{2}\int_0^{\pi/4} d\theta\right) = \frac{\pi-2}{2},$$

using a reduction of order formula.

4b Let I be the integral. With the substitution $t = 4\sec\theta$ we obtain

$$I = \frac{1}{16}\int \frac{\tan\theta}{\sec\theta|\tan\theta|}d\theta.$$

If $t > 4$: then $\sec\theta > 1$, implying $\theta \in [0, \pi/2)$ so that $|\tan\theta| = \tan\theta$, and hence

$$I = \frac{1}{16}\int \frac{\tan\theta}{\sec\theta\tan\theta}d\theta = \frac{1}{16}\int \frac{1}{\sec\theta}d\theta = \frac{1}{16}\int \cos\theta d\theta = \frac{1}{16}\sin\theta + C = \frac{\sqrt{t^2-16}}{16t} + C.$$

If $t < -4$: then $\sec\theta < 1$, implying $\theta \in (\pi/2, \pi]$ so that $|\tan\theta| = -\tan\theta$, and hence

$$I = -\frac{1}{16}\int \frac{\tan\theta}{\sec\theta\tan\theta}d\theta = -\frac{1}{16}\int \cos\theta d\theta = -\frac{1}{16}\sin\theta + C = -\frac{\sqrt{t^2-16}}{16t} + C.$$

Most generally we have

$$I = \begin{cases} -\dfrac{\sqrt{t^2-16}}{16t} + C_1, & \text{if } t < -4 \\ \dfrac{\sqrt{t^2-16}}{16t} + C_2, & \text{if } t > 4, \end{cases}$$

where C_1 and C_2 are each arbitrary constants (not necessarily equal).

5a With partial fraction decomposition integral becomes

$$\int_1^2 \left(\frac{1}{x} + \frac{4}{3x-2}\right)dx = \left[\ln|x| + \frac{4}{3}\ln|3x-2|\right]_1^2 = \frac{11}{3}\ln 2.$$

5b Perform a long division first, then apply partial fraction decomposition:

$$\begin{aligned}\int\left(x - \frac{9x^2-1}{x^3+9x}\right)dx &= \int\left(x + \frac{1/9}{x} - \frac{82x/9}{x^2+9}\right)dx \\ &= \frac{1}{2}x^2 + \frac{1}{9}\ln|x| - \frac{82}{9}\int\frac{x}{x^2+9}\,dx \\ &= \frac{1}{2}x^2 + \frac{1}{9}\ln|x| - \frac{41}{9}\ln(x^2+9) + C.\end{aligned}$$

6a Let $u = e^x$ first, and then let $u = \tan\theta$:

$$\int \frac{du}{u^2\sqrt{1+u^2}} = \int \frac{\sec\theta}{\tan^2\theta\sqrt{\sec^2\theta}}\,d\theta.$$

Since $\tan\theta = e^x > 0$ we have $\theta \in (0, \pi/2)$, and so $\sqrt{\sec^2\theta} = |\sec\theta| = \sec\theta$. With the substitution $w = \sin\theta$, integral becomes

$$\int \frac{\sec\theta}{\tan^2\theta}\,d\theta = \int \frac{\cos\theta}{\sin\theta}\,d\theta = \int \frac{1}{w^2}\,dw = -\frac{1}{w} + C = -\frac{\sqrt{1+e^{2x}}}{e^x} + C.$$

6b Apply integration by parts with $u = \tan^{-1} s$, $v' = s^2$:

$$\begin{aligned}\int s^2\tan^{-1} s\,ds &= \frac{s^3}{3}\tan^{-1} s - \frac{1}{3}\int \frac{s^3}{1+s^2}\,ds \\ &= \frac{s^3}{3}\tan^{-1} s - \frac{1}{3}\int \left(s - \frac{s}{1+s^2}\right) ds \\ &= \frac{s^3}{3}\tan^{-1} s - \frac{1}{6}s^2 + \frac{1}{6}\ln(1+s^2) + C.\end{aligned}$$

7a Integral diverges:

$$\lim_{t\to-\infty}\int_t^{-1} x^{-1/3}\,dx = \lim_{t\to-\infty}\left[\tfrac{3}{2}x^{2/3}\right]_t^{-1} = \lim_{t\to-\infty}\tfrac{3}{2}(1-t^{2/3}) = -\infty.$$

7b Let $u = \tan^{-1} t$, so integral $\int_0^\infty$ becomes

$$\int_0^\infty \frac{(\tan^{-1} t)^2}{t^2+1}\,dt = \lim_{r\to\infty}\int_0^{\tan^{-1} r} u^2\,du = \lim_{r\to\infty}\left[\frac{u^3}{3}\right]_0^{\tan^{-1} r} = \frac{1}{3}\lim_{r\to\infty}(\tan^{-1} r)^3 = \frac{\pi^3}{24}.$$

Since the integrand is an even function we also find that

$$\int_{-\infty}^0 \frac{(\tan^{-1} t)^2}{t^2+1}\,dt = \frac{\pi^3}{24}.$$

Therefore

$$\int_{-\infty}^\infty \frac{(\tan^{-1} t)^2}{t^2+1}\,dt = \int_{-\infty}^0 + \int_0^\infty = \frac{\pi^3}{24} + \frac{\pi^3}{24} = \frac{\pi^3}{12}.$$

7c First we evaluate

$$\int_0^3 \frac{dq}{\sqrt{9-q^2}} = \lim_{t\to3^-}\int_0^t \frac{dq}{\sqrt{9-q^2}} = \lim_{t\to3^-}\left[\sin^{-1}\frac{q}{3}\right]_0^t = \lim_{t\to3^-}\sin^{-1}\frac{t}{3} = \frac{\pi}{2}.$$

Since the integrand is an even function, we then obtain

$$\int_{-3}^3 \frac{dq}{\sqrt{9-q^2}} = 2\int_0^3 \frac{dq}{\sqrt{9-q^2}} = \pi.$$

8 For all $x \geq 0$ we find that

$$\frac{1}{e^x + x + 1} < \frac{1}{e^x},$$

and since

$$\int_0^\infty \frac{1}{e^x}\, dx = \lim_{t\to\infty} \int_0^t e^{-x}\, dx = \lim_{t\to\infty} \big[-e^{-x} \big]_0^t = \lim_{t\to\infty} (1 - e^{-t}) = 1,$$

the Comparison Theorem implies that the given integral converges.