Math 141 Exam #1 Key (Fall 2019)

1 The two inverses are

$$f^{-1}(y) = \sqrt{\frac{2-2y}{y}} \quad \text{and} \quad f^{-1}(y) = -\sqrt{\frac{2-2y}{y}}.$$

The first is the inverse of $f(x)$ for $x \geq 0$, while the second is the inverse for $x < 0$. The domain of both inverses is $(0, 1]$.

2 Since $f(1) = 3$ and $f'(x) = 3x^2 + 1$, the Inverse Function Theorem gives

$$(f^{-1})'(3) = (f^{-1})'(f(1)) = \frac{1}{f'(1)} = \frac{1}{4}.$$

3a $f'(x) = \dfrac{1}{x+1} - \dfrac{1}{x-1}.$

3b Quotient rule: $F'(x) = \dfrac{7e^{5x} + 72e^{6x}}{(e^{-x} + 12)^2}.$

3c $s'(t) = \dfrac{d}{dt} e^{\ln(8)\sec(2t)} = 2\ln 8 \cdot 8^{\sec 2t} \sec 2t \tan 2t$

3d $g'(x) = 2x^{\ln x - 1} \ln x$

3e $u'(x) = \dfrac{10}{(10x-1)\ln 10}.$

3f $h'(r) = (\ln r)^{\cos r} \left[\dfrac{\cos r}{r \ln r} - \sin r \cdot \ln(\ln r) \right]$

3g $q'(z) = \dfrac{10e^{10z}}{1 + e^{20z}}.$

3h $p'(x) = -12 \sinh^2 4x \cosh 4x.$

4 Here

$$y' = \frac{3x^2}{x^3 - 7},$$

so when $x = 2$ we have $y' = 12$. This is the slope of the tangent line through the point $(2, 0)$, so the equation is $y = 12x - 24$.

5a Letting $u = 8 - 3t$, we have

$$\int_2^4 \frac{2}{16 - 3t} dt = -\frac{2}{3} \left[\ln|16 - 3t| \right]_2^4 = -\frac{2}{3} \ln \frac{2}{5}.$$

5b Let $u = \ln(\ln x)$, so $\frac{1}{x \ln x} dx = du$ and we get

$$\int \frac{1}{u}\, du = \ln|u| + C = \ln|\ln(\ln x)| + C.$$

5c Let $u = \ln z$:

$$10 \int \frac{\log_2 z}{z}\, dz = 10 \int \frac{\ln z}{z \ln 2}\, dz = \frac{10}{\ln 2} \int u\, du = \frac{10}{\ln 2} \cdot \frac{u^2}{2} + C = \frac{5}{\ln 2}(\ln z)^2 + C.$$

5d Let $u = x^x$, so $du = x^x(1 + \ln x)dx$ and the integral becomes

$$\int_1^4 du = u\Big|_1^4 = 3.$$

5e Let $u = e^x$, so $du = e^x\, dx$ and the integral becomes

$$\int \frac{1}{u^2 + 4}\, du = \frac{1}{2} \tan^{-1} \frac{u}{2} + C = \frac{1}{2} \tan^{-1} \frac{e^x}{2} + C.$$

5f Let $u = \cosh 3y$ to get

$$\frac{1}{3} \int_1^{\cosh 3} u^3\, du = \frac{1}{12} \left[u^4\right]_1^{\cosh 3} = \frac{\cosh^4 3 - 1}{12}.$$

6 Length is

$$\int_0^a \sqrt{1 + \sinh^2 x}\, dx = \int_0^a \cosh x\, dx = \left[\sinh x\right]_0^a = \sinh a.$$

7a $\displaystyle \lim_{x \to 0} \frac{\sin^{-1} x}{x} = \lim_{x \to 0} \frac{1}{\sqrt{1 - x^2}} = 1.$

7b We have

$$\exp\left[\lim_{x \to 0} \frac{\ln(x + \cos x)}{x}\right] \overset{\text{LR}}{=} \exp\left(\lim_{x \to 0} \frac{1 - \sin x}{x + \cos x}\right) = \exp(1) = e.$$

7c We have

$$\begin{aligned} \lim_{x \to 0^+} (a^x - b^x)^x &= \exp\left[\lim_{x \to 0^+} \frac{\ln(a^x - b^x)}{1/x}\right] \overset{\text{LR}}{=} \exp\left[\ln \frac{b}{a} \lim_{x \to 0^+} \left(\frac{x^2}{a^x - b^x}\right)\right] \\ &\overset{\text{LR}}{=} \exp\left(\ln \frac{b}{a} \lim_{x \to 0^+} \frac{2x}{a^x \ln a - b^x \ln b}\right) \\ &= \exp(0) = 1. \end{aligned}$$