Math 141 Exam #4 Key (Fall 2018)

1 Approximate $\tan(-0.1)$ with the 3rd-order Taylor polynomial centered at 0. For x "near" 0 we have

$$\tan x \approx \sum_{n=0}^{3} \frac{\tan^{(n)}(0)}{n!}x^n = \tan(0) + \tan'(0)x + \frac{\tan''(0)}{2}x^2 + \frac{\tan'''(0)}{6}x^3.$$

Since $\tan' = \sec^2$, $\tan'' = 2\sec^2\tan$, $\tan''' = 2\sec^4 + 4\sec^2\tan^2$, we obtain

$$\tan x \approx x + \frac{1}{3}x^3,$$

and thus

$$\tan(-0.1) \approx -0.1 + \frac{1}{3}(-0.1)^3 = -\frac{301}{3000} = -0.1003333333...$$

The exact value is $\tan(-0.1) = -0.1003346721...$, and so the absolute error is about 1.339×10^{-6}.

2 Here $p_2(x) = 1 + x + x^2/2$ is the 2nd-order Taylor polynomial for $f(x) = e^x$ with center $x_0 = 0$. Let $I = [-\frac{1}{2}, \frac{1}{2}]$. Now, $f^{(3)}(x) = e^x$, and for all $t \in I$ we have

$$|f^{(3)}(t)| = e^t \leq e^{1/2}.$$

A theorem from the homework now implies that the remainder $R_2 = f - p_2$ is such that

$$|R_2(x)| \leq \frac{e^{1/2}|x|^3}{3!}$$

for all $x \in I$. In particular we find that

$$|f(x) - p_2(x)| \leq \frac{e^{1/2}(1/2)^3}{3!} = \frac{\sqrt{e}}{48} \approx 0.03435$$

for all $-\frac{1}{2} \leq x \leq \frac{1}{2}$.

3a Apply Ratio Test:

$$\lim_{n\to\infty}\left|\frac{x^{n+1}}{\sqrt{(n+1)^2+3}} \cdot \frac{\sqrt{n^2+3}}{x^n}\right| = |x| \lim_{n\to\infty}\left|\sqrt{\frac{n^2+3}{n^2+2n+4}}\right| = |x|.$$

Series converges if $|x| < 1$, so interval of convergence contains $(-1, 1)$. Check endpoints.

At $x = 1$: series becomes $\sum 1/\sqrt{n^2+3}$, and since

$$\frac{1}{\sqrt{n^2+3}} > \frac{1}{\sqrt{n^2}} = \frac{1}{n}$$

and the series $\sum 1/n$ is known to diverge, the series $\sum 1/\sqrt{n^2+3}$ diverges by the Direct Comparison Test.

At $x = -1$: series becomes $\sum(-1)^n/\sqrt{n^2+3}$, which can be shown to converge by the Alternating Series Test.

Therefore the original series has interval of convergence $[-1, 1)$.

3b Apply the Root Test:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left|1+\frac{1}{n}\right| |x+2| = |x+2| \lim_{n\to\infty} \left(1+\frac{1}{n}\right) = |x+2|.$$

Series converges if $|x+2| < 1$, so interval of convergence contains $(-3,-1)$. Check endpoints.

At $x=-1$: Series becomes

$$\sum \left(1+\frac{1}{n}\right)^n,$$

and since $\left(1+\frac{1}{n}\right)^n \to e$ as $n\to\infty$, the series diverges by the Divergence Test.

At $x=-3$: Series becomes

$$\sum \left(1+\frac{1}{n}\right)^n (-1)^n = \sum (-1)^n \left(1+\frac{1}{n}\right)^n,$$

which also diverges by the Divergence Test.

Therefore the original series has interval of convergence $(-3,-1)$.

3c Apply Ratio Test, using L'Hôpital's Rule to find the limit:

$$\lim \left|\frac{x^{n+1}\ln(n+1)}{x^n \ln n}\right| = |x| \lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} \overset{\text{LR}}{=} |x| \lim_{n\to\infty} \frac{1/(n+1)}{1/n} = |x|.$$

Thus the series converges if $|x| < 1$, so interval of convergence contains $(-1,1)$. At the endpoints we obtain either the series $\sum \ln n$ or $\sum (-1)^n \ln n$, both of which diverge by the Divergence Test. Therefore the original series has interval of convergence $(-1,1)$.

4 We have

$$f(x) = \sum_{n=0}^{\infty} e^{-nx} = \sum_{n=0}^{\infty} (e^{-x})^n = \frac{1}{1-e^{-x}} = \frac{e^x}{e^x-1}.$$

5 Using the binomial series,

$$(1+x^2)^{-1/3} = \sum_{n=0}^{\infty} \binom{-1/3}{n} x^{2n} \approx 1 - \frac{1}{3}x^2 + \frac{2}{9}x^4 - \frac{14}{81}x^6.$$

6 We have

$$\int_0^{0.1} \frac{\ln(1+x)}{x}\,dx = \int_0^{0.1} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^{n-1}}{n}\,dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \int_0^{0.1} x^{n-1}dx$$
$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(0.1)^n}{n^2} = 0.1 - \frac{(0.1)^2}{2^2} + \frac{(0.1)^3}{3^2} - \cdots.$$

Since $(0.1)^3/3^2 < 10^{-5}$, the estimate

$$\int_0^{0.1} \frac{\ln(1+x)}{x}\,dx \approx 0.1 - \frac{(0.1)^2}{2^2} = 0.0975$$

will have an absolute error less than 10^{-5}.

7 Use the identity $\tan^2 + 1 = \sec^2$ to find that $y^2 + 1 = \sec^2 t$, so that

$$x = \sec^2 t - 1 = (y^2 + 1) - 1 = y^2.$$

Thus we have $x = y^2$ with domain $y \in (-\infty, \infty)$, recalling that $y = \tan t$ for $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

8 Parametric equations: $x = \cos t + 2$, $y = \sin t + 3$. Answers can vary.

9 Rewrite as $r \sin\theta = e^{r\cos\theta}$, which in rectangular coordinates becomes $y = e^x$.

10 The top half of the inner loop is traced when $\theta \in [0, \pi/3]$. Using the identity $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$, the area is

$$\mathcal{A} = 2\int_0^{\pi/3} \frac{(\cos\theta - 1/2)^2}{2}\,d\theta = \int_0^{\pi/3} \left(\frac{\cos 2\theta}{2} - \cos\theta + \frac{3}{4}\right) d\theta = \frac{\pi}{4} - \frac{3\sqrt{3}}{8}.$$