Math 141 Exam #4 Key (Fall 2017)

1 We have

$$\ln(1.1) = \ln(1+0.1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(0.1)^n}{n} \approx \sum_{n=1}^{4} \frac{(-1)^{n+1}(0.1)^n}{n}$$

$$= 0.1 - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} = 0.095308\overline{3}.$$

2a Apply Ratio Test:

$$\lim_{n\to\infty} \left| \frac{2^{2(n+1)}x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{2^{2n}x^n} \right| = |x| \lim_{n\to\infty} \frac{4n^2}{(n+1)^2} = 4|x|.$$

Series converges if $|x| < 1/4$, so interval of convergence contains $(-1/4, 1/4)$. Check endpoints.

At $x = 1/4$: series becomes $\sum 1/n^2$, a convergent p-series. At $x = -1/4$: series becomes $\sum(-1)^n/n^2$, which converges by the Alternating Series Test.

Interval of convergence is $\left[-\frac{1}{4}, \frac{1}{4}\right]$.

2b Ratio Test:

$$\lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{(n+1)\cdot 5^{n+1}} \cdot \frac{n \cdot 5^n}{(x-2)^n} \right| = |x-2| \lim_{n\to\infty} \frac{n}{5n+5} = \frac{|x-2|}{5}.$$

Series converges if $|x-2| < 5$, so interval of convergence contains $(-3, 7)$. Check endpoints.

At $x = 7$ series becomes $\sum 1/n$, which diverges. At $x = -3$ series becomes $\sum(-1)^n/n$, which converges by the Alternating Series Test.

Interval of convergence is $[-3, 7)$.

2c Ratio Test:

$$\lim \left| \frac{(n+1)!(x-1)^{n+1}}{n!(x-1)^n} \right| = \lim_{n\to\infty} (n+1)|x-1| = \begin{cases} \infty, & x \neq 1 \\ 0, & x = 1. \end{cases}$$

The series only converges at $\{1\}$.

3 Use the geometric series:

$$\frac{2x^2}{1+x^3} = 2x^2 \cdot \frac{1}{1-(-x^3)} = 2x^2 \sum_{n=0}^{\infty} (-x^3)^n = \sum_{n=0}^{\infty} (-1)^n 2x^{3n+2}.$$

Interval of convergence is $|-x^3| < 1$, and hence $(-1, 1)$.

4 Using the binomial series,

$$f(x) = \sum_{n=0}^{\infty} \binom{1/4}{n} x^n = 1 + \frac{1}{4}x - \frac{3}{32}x^2 + \frac{7}{128}x^3 + \cdots.$$

Interval of convergence is $(-1, 1)$.

5 From the table provided we have $e^x = \sum_{n=0}^{\infty} x^n/n!$ for all $x \in (-\infty, \infty)$, and so

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$$

for all x. Now,

$$\int \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n} \right) dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} x^{2n+1} + c$$

for all x and arbitrary constant c. Thus, by the Fundamental Theorem of Calculus,

$$\int_0^{1/3} e^{-x^2} dx = \int_0^{1/3} \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n} \right) dx = \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} x^{2n+1} \right]_0^{1/3} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} \left(\frac{1}{3} \right)^{2n+1}.$$

We have arrived at an alternating series $\sum (-1)^n b_n$ with

$$b_n = \frac{1}{n!(2n+1)} \left(\frac{1}{3} \right)^{2n+1}$$

for $n \geq 0$. The first few b_n values are

$$b_0 = \tfrac{1}{3}, \quad b_1 = \tfrac{1}{81}, \quad b_2 = \tfrac{1}{2430}, \quad b_3 = \tfrac{1}{91{,}854}, \quad b_4 = \tfrac{1}{4{,}251{,}528},$$

so by the Alternating Series Estimation Theorem the approximation

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} \left(\frac{1}{3} \right)^{2n+1} \approx b_0 - b_1 + b_2 - b_3 \approx 0.3213882901$$

will have an absolute error that is less than $b_4 \approx 2.35 \times 10^{-7} < 10^{-6}$. Hence the approximation

$$\int_0^{1/3} e^{-x^2} dx \approx \frac{1}{3} - \frac{1}{81} + \frac{1}{2430} - \frac{1}{91,854}$$

has an absolute error less than 10^{-6}.

6 Use the identity $\tan^2 + 1 = \sec^2$ to find that $y^2 + 1 = \sec^2 t$, so that

$$x = \sec^2 t - 1 = (y^2 + 1) - 1 = y^2.$$

Thus we have $x = y^2$ with domain $y \in (-\infty, \infty)$, recalling that $y = \tan t$ for $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

7 The set-up is thus:

$$(x, y) = \left(1 - \tfrac{1}{3}t\right)(3, -4) + \tfrac{1}{3}t\,(2, 0)$$

for $0 \leq t \leq 3$. Equivalently we may write

$$(x, y) = \left(-\tfrac{1}{3}t + 3, \tfrac{4}{3}t - 4\right), \quad t \in [0, 3]$$

8 Rewrite as $4r\cos\theta + 3r\sin\theta = 2$, and so obtain $4x + 3y = 2$.

9 Using the identity $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$, area is

$$\begin{aligned}
\mathcal{A} &= \frac{1}{2}\int_0^{2\pi} (2+\cos\theta)^2\,d\theta = \frac{1}{2}\int_0^{2\pi} (4+4\cos\theta+\cos^2\theta)\,d\theta \\
&= 2\int_0^{2\pi} d\theta + 2\int_0^{2\pi} \cos\theta\,d\theta + \frac{1}{4}\int_0^{2\pi} (1+\cos 2\theta)\,d\theta \\
&= 4\pi + 2\big[\sin\theta\big]_0^{2\pi} + \frac{1}{4}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{2\pi} = 4\pi + 0 + \frac{\pi}{2} = \frac{9\pi}{2}.
\end{aligned}$$