Math 141 Exam #3 Key (Fall 2017)

1a Limit is 1/3.

1b We have

$$\lim_{n\to\infty} \ln[n\sin(1/n)] = \lim_{n\to\infty} \ln\left(\frac{\sin(1/n)}{1/n}\right) = \ln(1) = 0.$$

2 Since $-\pi/2 < \tan^{-1} n < \pi/2$ for any integer n, we have

$$-\frac{2\pi}{n^4+1} < \frac{4\tan^{-1} n}{n^4} < \frac{2\pi}{n^4+1}$$

for all n, and since

$$\lim_{n\to\infty} \frac{2\pi}{n^4+1} = 0,$$

the Squeeze Theorem implies that

$$\lim_{n\to\infty} \frac{4\tan^{-1} n}{n^4} = 0.$$

3 Reindex to obtain

$$\sum_{k=2}^{\infty} \frac{5}{3^k} = \sum_{k=0}^{\infty} \frac{5}{3^{k+2}} = \sum_{k=0}^{\infty} \frac{5}{9}\left(\frac{1}{3}\right)^k = \frac{5/9}{1-1/3} = \frac{5}{6}.$$

4 The nth partial sum is

$$\begin{aligned} s_n &= (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + \cdots + [\ln n - \ln(n-1)] + [\ln(n+1) - \ln n] \\ &= -\ln 1 + \ln(n+1) = \ln(n+1), \end{aligned}$$

and so

$$\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right) = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \ln(n+1) = \infty.$$

That is, the series diverges.

5 Find the smallest integer value of n for which $\frac{1}{2n^4} < \frac{1}{1000}$. Since

$$\frac{1}{2n^4} < \frac{1}{1000} \quad \Rightarrow \quad n^4 > 500,$$

and $4^4 < 500$ while $5^4 > 500$, the estimation

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n^4} \approx \sum_{n=1}^{4} \frac{(-1)^n}{2n^4} = -\frac{1}{2} + \frac{1}{32} - \frac{1}{162} + \frac{1}{512}$$

has absolute error less than 10^{-3}.

6a Since $5/(10 - e^{-n}) \to 1/2 \neq 0$ as $n \to \infty$, the series diverges by the Divergence Test.

6b Making the substitution $u = x^3$, we have

$$\int_1^\infty x^2 e^{-x^3} dx = \frac{1}{3}\int_1^\infty e^{-u} du = \frac{1}{3}\lim_{t\to\infty}\big[-e^{-u}\big]_1^t = -\frac{1}{3}\lim_{t\to\infty}(e^{-t} - e^{-1}) = \frac{1}{3e},$$

and so the series converges by the Integral Test.

6c We have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(\frac{[(n+1)!]^2}{[2(n+1)]!}\cdot\frac{(2n)!}{(n!)^2}\right) = \lim_{n\to\infty}\frac{(n+1)^2}{(2n+1)(2n+2)} = \frac{1}{4} \in [0,1),$$

and so the series converges by the Ratio Test.

6d We have

$$\lim_{n\to\infty}\sqrt[n]{\frac{n^2}{2^n}} = \frac{1}{2}\lim_{n\to\infty} n^{2/n} = \frac{1}{2}\lim_{n\to\infty} e^{\ln n^{2/n}} = \frac{1}{2}\exp\left(\lim_{n\to\infty}\frac{2\ln n}{n}\right) = \frac{1}{2}\exp(0) = \frac{1}{2} < 1,$$

so the series converges by the Root Test.

6e Since

$$0 \leq \frac{\cos^2 n}{n^2+1} \leq \frac{1}{n^2+1} < \frac{1}{n^2}$$

and $\sum \frac{1}{n^2}$ is a convergent p-series, the series converges by the Comparison Test.

6f Since

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(\frac{2\cdot 6\cdot 10\cdots[2+4(n+1)]}{5\cdot 8\cdot 11\cdots[5+3(n+1)]}\cdot\frac{5\cdot 8\cdot 11\cdots(5+3n)}{2\cdot 6\cdot 10\cdots(2+4n)}\right)$$

$$= \lim_{n\to\infty}\frac{2+4(n+1)}{5+3(n+1)} = \frac{4}{3} > 1,$$

the series diverges by the Ratio Test.

7a Since $1/\ln^2 n$ is a decreasing sequence of nonnegative values such that $1/\ln^2 n \to 0$ as $n \to \infty$, the series converges by the Alternating Series Test. Does not converge absolutely, however, and so the series is conditionally convergent.

7b Since

$$\lim_{n\to\infty}\left(1 + \frac{2}{n}\right)^n \neq 0,$$

the series diverges by the Divergence Test.