Math 141 Exam #2 Key (Fall 2017)

1 Let $u = \ln t$, $v' = t^3$, so $u' = 1/t$, $v = t^4/4$. Then

$$\int t^3 \ln t\, dt = \frac{1}{4}t^4 \ln t - \int \frac{1}{4}t^3\, dt = \frac{1}{4}t^4 \ln t - \frac{1}{16}t^4 + c.$$

2 Let $u = x$, $v' = \sec^2 x$, so $u' = 1$, $v = \tan x$. Then

$$\int x \sec^2 x\, dx = x \tan x - \int \tan x\, dx = x \tan x - \ln|\sec x| + c.$$

3 Let $u = \sec^{-1} x$, $v' = x$, so $u' = \frac{1}{x\sqrt{x^2-1}}$, $v = x^2/2$. If I is the integral, then

$$I = \left[\frac{1}{2}x^2 \sec^{-1} x\right]_{2/\sqrt{3}}^{2} - \int_{2/\sqrt{3}}^{2} \frac{x}{2\sqrt{x^2-1}}\, dx.$$

With the substitution $u = x^2 - 1$ we obtain

$$I = 2\sec^{-1} 2 - \frac{2}{3}\sec^{-1}\frac{2}{\sqrt{3}} - \frac{1}{4}\int_{1/3}^{3} \frac{1}{\sqrt{u}}\, du = \frac{5}{9}\pi - \frac{1}{\sqrt{3}}.$$

4 With the substitution $u = \sin x$, the integral becomes

$$\int_0^{\pi/2} \sin^{5/2} x(1 - \sin^2 x)\cos x\, dx = \int_0^1 u^{5/2}(1 - u^2)\, du = \frac{8}{77}.$$

5 Using the reduction of order formula, the integral becomes

$$\left[\frac{\sec^2\theta \tan\theta}{3}\right]_0^{\pi/4} + \frac{2}{3}\int_0^{\pi/4} \sec^2\theta\, d\theta = \frac{2}{3} + \frac{2}{3}\big[\tan\theta\big]_0^{\pi/4} = \frac{4}{3}.$$

6 Let $y = \frac{1}{3}\tan\theta$, so $\frac{1}{9} + y^2 = \frac{1}{9}\sec^2\theta$ and $dy = \frac{1}{3}\sec^2\theta\, d\theta$. Integral becomes

$$\int_0^{1/3} \frac{1}{\sqrt{1/9 + y^2}}\, dy = \int_0^{\pi/4} \frac{\frac{1}{3}\sec^2\theta}{\sqrt{\frac{1}{9}\sec^2\theta}}\, d\theta = \int_0^{\pi/4} \sec\theta\, d\theta = \big[\ln|\sec\theta + \tan\theta|\big]_0^{\pi/4} = \ln(\sqrt{2} + 1).$$

7 Let $t = 5\sin\theta$, so $dt = 5\cos\theta\, d\theta$. Using a reduction of order formula, the integral I becomes

$$I = \int \sqrt{25 - 25\sin^2\theta} \cdot 5\cos\theta\, d\theta = 25\int \cos^2\theta\, d\theta = 25\left(\frac{\cos\theta\sin\theta}{2} + \frac{1}{2}\int d\theta\right) + c$$

$$= 25\left(\frac{1}{2} \cdot \frac{\sqrt{25 - t^2}}{5} \cdot \frac{t}{5} + \frac{1}{2}\sin^{-1}\frac{t}{5}\right) + c = \frac{t\sqrt{25 - t^2}}{2} + \frac{25}{2}\sin^{-1}\frac{t}{5} + c.$$

8a By partial fractions technique:

$$\frac{12r}{(r-4)^2} = \frac{A}{r-4} + \frac{B}{(r-4)^2} \quad \Rightarrow \quad 12r = A(r-4) + B \quad \Rightarrow \quad A = 12,\ B = 48.$$

So,

$$\int \frac{12r}{(r-4)^2}\,dr = \int \left(\frac{12}{r-4} + \frac{48}{(r-4)^2} \right) dr = 12 \ln |r-4| - \frac{48}{r-4} + c.$$

8b We have

$$\frac{x+1}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2},$$

which yields $A = -3/4$, $B = -1/2$, $C = 3/4$. Then

$$\int \frac{x+1}{x^2(x-2)}\,dx = \int \left(\frac{-3/4}{x} - \frac{1/2}{x^2} + \frac{3/4}{x-2} \right) dx = \frac{3}{4} \ln \left| \frac{x-2}{x} \right| + \frac{1}{2x} + c.$$

9a Making the substitution $u = \ln y$,

$$\int_2^\infty \frac{dy}{y \ln y} = \lim_{t\to\infty} \int_2^t \frac{dy}{y \ln y} = \lim_{t\to\infty} \int_{\ln 2}^{\ln t} \frac{1}{u}\,du = \lim_{t\to\infty} \big[\ln|\ln t| - \ln|\ln 2| \big] = \infty.$$

The integral diverges.

9b Since

$$\lim_{t\to 1^-} \int_0^t \frac{1}{(x-1)^{2/3}}\,dx = \lim_{t\to 1^-} \left[3(x-1)^{1/3} \right]_0^t = \lim_{t\to 1^-} \left[3(t-1)^{1/3} + 3 \right] = 3$$

and

$$\lim_{t\to 1^+} \int_t^3 \frac{1}{(x-1)^{2/3}}\,dx = \lim_{t\to 1^+} \left[3(x-1)^{1/3} \right]_t^3 = \lim_{t\to 1^+} \left[3\sqrt[3]{2} - 3(t-1)^{1/3} \right] = 3\sqrt[3]{2},$$

the integral $\int_0^3$ is convergent. In particular,

$$\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx = 3 + 3\sqrt[3]{2}.$$

10 Volume is

$$\int_0^\infty \pi (e^{-x})^2\,dx = \lim_{t\to\infty} \pi \int_0^t e^{-2x}\,dx = -\frac{\pi}{2} \lim_{t\to\infty} \left[e^{-2x} \right]_0^t = \frac{\pi}{2}.$$