Math 141 Exam #1 Key (Fall 2017)

1a From $y^4 = 4x^2$ we obtain $|y| = \sqrt[4]{4x^2} = \sqrt{2|x|}$. Now, $f_1(x) = \sqrt{2x}$ for $x \in [0, \infty)$, $f_2(x) = \sqrt{-2x}$ for $x \in (-\infty, 0]$, $f_3(x) = -\sqrt{-2x}$ for $x \in (-\infty, 0]$, and $f_4(x) = -\sqrt{2x}$ for $x \in [0, \infty)$.

1b Let $y = f_1(x)$, so that $y = \sqrt{2x}$ for $x \geq 0$. Solving for x yields $x = y^2/2$, and thus $f_1^{-1}(y) = y^2/2$ for $y \geq 0$. Replacing y with x (a formal maneuver) then gives $f_1^{-1}(x) = x^2/2$ for $x \geq 0$. Thus $y = f_1^{-1}(x)$ is written $y = x^2/2$ (valid for $x \geq 0$, though this information is not asked for). Similarly: $f_2^{-1}(x) = -x^2/2$, $f_3^{-1}(x) = -x^2/2$, and $f_4^{-1}(x) = x^2/2$.

2 We have $f'(x) = 1 - \sin x$, so $f'(x) > 0$ for all $x \in (0, \pi/2) \cup (\pi/2, 2\pi)$, with $f'(\pi/2) = 0$. This is enough to show that f is increasing on $(0, 2\pi)$, hence is one-to-one on $(0, 2\pi)$, and therefore has on inverse on $(0, 2\pi)$. Now, since $f(\pi) = -1$, by the Inverse Function Theorem we have

$$(f^{-1})'(-1) = (f^{-1})'(f(\pi)) = \frac{1}{f'(\pi)} = \frac{1}{1 - \sin \pi} = 1.$$

3a Quotient rule:

$$y' = \frac{(\ln x + 1)(1/x) - (\ln x)(1/x)}{(\ln x + 1)^2} = \frac{1}{x(\ln x + 1)^2}.$$

3b Chain rule: $f'(x) = \cos(\cos e^x) \cdot (-\sin e^x) \cdot e^x$.

3c We have

$$\begin{aligned} g'(x) &= \frac{d}{dx}\left(e^{(x-1)\ln(\tan x)}\right) = e^{(x-1)\ln(\tan x)} \frac{d}{dx}\left[(x-1)\ln(\tan x)\right] \\ &= (\tan x)^{x-1}\left[\frac{(x-1)\sec^2 x}{\tan x} + \ln(\tan x)\right]. \end{aligned}$$

3d We have

$$h'(t) = \frac{d}{dt}\left(e^{t^{10}\ln t}\right) = e^{t^{10}\ln t}\frac{d}{dt}(t^{10}\ln t) = t^{(t^{10})}(t^9 + 10t^9 \ln t).$$

3e Use algebra to find that

$$y = \frac{4\ln(1 - x^3)}{\ln 3}.$$

Now,

$$y' = \frac{12x^2}{(x^3 - 1)\ln 3}.$$

3f By the Chain Rule:

$$\varphi'(u) = -\frac{1}{|4u-4|\sqrt{(4u-4)^2-1}} \cdot (4u-4)' = -\frac{1}{|u-1|\sqrt{(4u-4)^2-1}}.$$

3g $y' = 2x\cosh^3(5x) + 15x^2\cosh^2(5x)\sinh(5x)$.

4 Here $y'(x) = 2e^x$, so slope of the tangent line at $x = \ln 3$ is $y'(\ln 3) = 2e^{\ln 3} = 6$. Also the tangent line contains the point $(\ln 3, 5)$ since $y(\ln 3) = 2e^{\ln 3} - 1 = 5$. Equation of the line:

$$y = 6x + (5 - 6\ln 3).$$

5a Let $u = x^6$, so

$$\int x^5 e^{x^6}\,dx = \int e^u du = \frac{1}{6}e^u + c = \frac{1}{6}e^{x^6} + c.$$

5b Let $u = \ln(\ln x)$, so $du = \frac{1}{x\ln x}dx$, and the integral becomes

$$\int \frac{1}{u}du = \ln|u| + c = \ln\big|\ln(\ln x)\big| + c.$$

5c Let $u = \sin x$, so $du = \cos x\,dx = \frac{1}{\sec x}dx$. Integral becomes

$$\int e^u du = e^u + c = e^{\sin x} + c.$$

6a Letting $u = e^{t/2} + 1$, so integral becomes

$$\int_{1/e+1}^{e+1} \frac{2}{u}\,du = 2\ln|u|\Big]_{1/e+1}^{e+1} = 2\ln\left(\frac{e+1}{e^{-1}+1}\right) = 2.$$

6b Use a given formula:

$$\frac{1}{2}\int_0^{3/2} \frac{1}{\sqrt{9-x^2}}\,dx = \frac{1}{2}\left[\sin^{-1}\left(\frac{x}{3}\right)\right]_0^{3/2} = \frac{\pi}{12}.$$

6c Let $u = \cosh 4x$, so integral becomes

$$\frac{1}{4}\int_1^{\cosh 4} u^3\,du = \frac{1}{4}\left[\frac{1}{4}u^4\right]_1^{\cosh 4} = \frac{\cosh^4 4 - 1}{16}.$$

7 We have

$$\lim_{x\to\infty} e^{\frac{\ln(x^3+1)}{\ln x}} = \exp\left(\lim_{x\to\infty} \frac{\ln(x^3+1)}{\ln x}\right) \overset{\text{LR}}{=} \exp\left(\lim_{x\to\infty} \frac{3x^2/(x^3+1)}{1/x}\right) = \exp\left(\lim_{x\to\infty} \frac{3x^3}{x^3+1}\right) = e^3.$$