Math 141 Exam #4 Key (Fall 2016)

1 Let $f(x) = \sqrt[5]{x}$. The 3rd-order Taylor polynomial centered at 32 for f is

$$\begin{aligned} P_3(x) &= f(32) + f'(32)(x-32) + \frac{f''(32)}{2}(x-32)^2 + \frac{f'''(32)}{6}(x-32)^3 \\ &= 2 + \frac{1}{5}(32)^{-4/5}(x-32) - \frac{2}{25}(32)^{-9/5}(x-32)^2 + \frac{19}{375}(32)^{-14/5}(x-32)^3 \\ &= 2 + \tfrac{1}{80}(x-32) - \tfrac{1}{6400}(x-32)^2 + \tfrac{19}{6{,}144{,}000}(x-32)^3 \end{aligned}$$

and so

$$\sqrt[5]{31} = f(31) \approx P_3(31) = 2 - \tfrac{1}{80} - \tfrac{1}{6400} - \tfrac{19}{6{,}144{,}000} \approx 1.987340658$$

(Note this is quite close to the actual value of 1.987340755....)

2a Clearly the series converges when $x = 0$. Assuming $x \neq 0$, we find that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^3x^{4(n+1)}}{(n+1)!}\cdot\frac{n!}{n^3x^{4n}}\right| = \lim_{n\to\infty}\frac{(n+1)^2x^4}{n^3} = 0$$

for all x, and so by the Ratio Test the series converges on $(-\infty,\infty)$. There are no endpoints to consider here.

2b Since

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^nx^{n+1}}{(n+1)^3}\cdot\frac{n^3}{(-1)^{n-1}x^n}\right| = \lim_{n\to\infty}\frac{n^3|x|}{(n+1)^3} = |x|,$$

by the Ratio Test the series converges if $|x| < 1$, which implies $x \in (-1,1)$.

When $x = 1$ the series becomes

$$\sum\frac{(-1)^{n-1}}{n^3},$$

which converges by the Alternating Series Test. When $x = -1$ the series becomes

$$\sum\frac{(-1)^{2n-1}}{n^3} = -\sum\frac{1}{n^3},$$

which is a convergent p-series. The interval of convergence is therefore $[-1,1]$.

2c Since

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-2)^{n+1}(x-1)^{n+1}}{\sqrt[4]{n+1}}\cdot\frac{\sqrt[4]{n}}{(-2)^n(x-1)^n}\right| = \lim_{n\to\infty}2|x-1|\sqrt[4]{\frac{n}{n+1}} = 2|x-1|,$$

by the Ratio Test the series converges if $2|x-1| < 1$, which implies $x \in \left(\frac{1}{2},\frac{3}{2}\right)$.

When $x = 1/2$ the series becomes

$$\sum\frac{1}{\sqrt[4]{n}} = \sum\frac{1}{n^{1/4}},$$

which is a divergent p-series. When $x = 3/2$ the series becomes

$$\sum \frac{(-1)^n}{\sqrt[4]{n}},$$

which converges by the Alternating Series Test. The interval of convergence is therefore $\left(\frac{1}{2}, \frac{3}{2}\right]$.

3 From the table provided we have

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

for $|x| < 1$. Differentiate this once to obtain

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}. \tag{1}$$

The given series can be manipulated to resemble this, as follows:

$$\sum_{n=1}^{\infty} (-1)^n \frac{n x^{n+1}}{6^n} = x \sum_{n=1}^{\infty} n\left(-\frac{x}{6}\right)^n = -\frac{x^2}{6} \sum_{n=1}^{\infty} n\left(-\frac{x}{6}\right)^{n-1}. \tag{2}$$

Using (1) with x replaced by $-x/6$, from (2) we now obtain

$$\sum_{n=1}^{\infty} (-1)^n \frac{n x^{n+1}}{6^n} = -\frac{x^2}{6} \cdot \frac{1}{(1+x/6)^2} = -\frac{6x^2}{(x+6)^2}.$$

4a We have

$$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots$$

4b In summation notation:

$$\ln x = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(x-1)^k}{k}.$$

4c The Ratio Test may be used to determine that the series converges on $(0, 2)$, and then, testing the endpoints 0 and 2, we find that the interval of convergence if $(0, 2]$. The radius of convergence is $R = 1$.

5 From the table provided we have the binomial series

$$\frac{1}{\sqrt{1+x^6}} = (1+x^6)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} x^n$$

for all $x \in (-1,1)$, and so

$$\begin{aligned}
\int_0^{0.5} \frac{1}{\sqrt{1+x^6}}\,dx &= \int_0^{0.5} \left[\sum_{n=0}^{\infty} \binom{-1/2}{n} x^n\right] dx = \sum_{n=0}^{\infty} \left[\int_0^{0.5} \binom{-1/2}{n} x^n\,dx\right] \\
&= \sum_{n=0}^{\infty} \left[\binom{-1/2}{n} \frac{x^{n+1}}{n+1}\right]_0^{0.5} = \sum_{n=0}^{\infty} \binom{-1/2}{n} \frac{(0.5)^{n+1}}{n+1} \\
&= \sum_{n=0}^{\infty} \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\cdots(-\frac{1}{2}-(n-1))}{(n+1)!} \left(\frac{1}{2}\right)^{n+1} \\
&= \frac{1}{2} + \sum_{n=1}^{\infty} (-1)^n \frac{(1)(3)(5)\cdots(2n-1)}{(n+1)!} \left(\frac{1}{2}\right)^{2n+1}.
\end{aligned}$$

This is an alternating series with $b_0 = \frac{1}{2}$ and

$$b_n = \frac{(1)(3)(5)\cdots(2n-1)}{(n+1)!} \left(\frac{1}{2}\right)^{2n+1}$$

for $n \geq 1$. We evaluate b_n values: $b_0 = \frac{1}{2}$, $b_1 = \frac{1}{16}$, $b_2 = \frac{1}{64}$, $b_3 = \frac{5}{1024}$, $b_4 = \frac{7}{4096} \approx 1.7 \times 10^{-3}$, and finally $b_5 \approx 6.41 \times 10^{-4}$. Thus b_5 is the first value that is less than 10^{-3}, and by the Alternating Series Estimation Theorem the approximation

$$\frac{1}{2} + \sum_{n=1}^{\infty} (-1)^n \frac{(1)(3)(5)\cdots(2n-1)}{(n+1)!} \left(\frac{1}{2}\right)^{2n+1} \approx b_0 - b_1 + b_2 - b_3 + b_4 \approx 0.4500$$

will have an absolute error that is less than 10^{-3}.

6 Here

$$x = \frac{3}{t+5} - 2 \quad \Rightarrow \quad t = \frac{3}{x+2} - 5,$$

and so

$$y = t + 1 = \left(\frac{3}{x+2} - 5\right) + 1 = \frac{3}{x+2} - 4.$$

That is,

$$f(x) = \frac{3}{x+2} - 4,$$

and from $t \in [0, 10]$ we see that the domain of f (i.e. the attainable values of x) will be the closed interval with endpoints

$$\frac{3}{0+5} - 2 = -\frac{7}{5} \quad \text{and} \quad \frac{3}{10+5} - 2 = -\frac{9}{5}.$$

That is, $\text{Dom}(f) = \left[-\frac{9}{5}, -\frac{7}{5}\right]$.

7 There are many possible parametrizations, but one good one is

$$(x(t), y(t)) = (-1, 0)(1 - t) + (0, 5)t = (t - 1, 5t)$$

for $t \in (-\infty, \infty)$.

8 It helps to multiply by r to get

$$r^2 = 2r \sin\theta + 2r \cos\theta.$$

Then, since $x = r\cos\theta$, $y = r\sin\theta$, and $r^2 = x^2 + y^2$, we obtain

$$x^2 + y^2 = 2y + 2x.$$

We can improve on this: from $(x^2 - 2x) + (y^2 - 2y) = 0$ we obtain

$$(x - 1)^2 + (y - 1)^2 = 2,$$

which is seen to be the equation of a circle centered at $(1, 1)$ with radius $\sqrt{2}$.

9 With $f(\theta) = 4\cos\theta$ and $f'(\theta) = -4\sin\theta$, the slope is

$$\frac{f'(\pi/3)\sin(\pi/3) + f(\pi/3)\cos(\pi/3)}{f'(\pi/3)\cos(\pi/3) - f(\pi/3)\sin(\pi/3)} = \frac{1}{\sqrt{3}}.$$