Math 141 Exam #3 Key (Fall 2016)

1a Recurrence relation:

$$a_{n+1} = (-1)^n(|a_n| + 1), \quad a_1 = 1.$$

1b Explicit formula:

$$a_n = (-1)^{n+1} n, \quad n \geq 1.$$

2a Since $\tan^{-1} n \to \pi/2$ as $n \to \infty$, we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\tan^{-1} n}{n} = \frac{\lim_{n\to\infty} \tan^{-1} n}{\lim_{n\to\infty} n} = 0.$$

2b Using L'Hôpital's Rule where indicated, we find that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n \ln\left(\frac{3n+1}{3n-1}\right) = \lim_{n\to\infty} \frac{\ln(3n+1) - \ln(3n-1)}{1/n} \overset{\text{LR}}{=} \lim_{n\to\infty} \frac{\frac{3}{3n+1} - \frac{3}{3n-1}}{-\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \left(\frac{3n^2}{3n-1} - \frac{3n^2}{3n+1}\right) = \lim_{n\to\infty} \frac{6n^2}{9n^2-1} = \frac{2}{3}.$$

3 We have

$$0.0\overline{213} = \frac{213}{10^4} + \frac{213}{10^7} + \frac{213}{10^{10}} + \cdots = \sum_{k=0}^{\infty} \frac{213}{10^{3k+4}}.$$

4 We have

$$s_n = \sum_{k=1}^{n} \left(\frac{1}{\sqrt{k+1}} - \frac{1}{\sqrt{k+3}}\right)$$

$$= \left(\tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{4}}\right) + \left(\tfrac{1}{\sqrt{3}} - \tfrac{1}{\sqrt{5}}\right) + \left(\tfrac{1}{\sqrt{4}} - \tfrac{1}{\sqrt{6}}\right) + \cdots + \left(\tfrac{1}{\sqrt{n-1}} - \tfrac{1}{\sqrt{n+1}}\right) + \left(\tfrac{1}{\sqrt{n}} - \tfrac{1}{\sqrt{n+2}}\right) + \left(\tfrac{1}{\sqrt{n+1}} - \tfrac{1}{\sqrt{n+3}}\right)$$

$$= \tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{3}} - \tfrac{1}{\sqrt{n+2}} - \tfrac{1}{\sqrt{n+3}}.$$

From this we see that

$$\sum_{k=1}^{\infty} \left(\frac{1}{\sqrt{k+1}} - \frac{1}{\sqrt{k+3}}\right) = \lim_{n\to\infty} s_n = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}}.$$

5 We have

$$\int_0^{\infty} \frac{10}{x^2+9} dx = 10 \lim_{t\to\infty} \int_0^t \frac{1}{x^2+3^2} dx = 10 \lim_{t\to\infty} \left[\frac{1}{3} \tan^{-1}\left(\frac{x}{3}\right)\right]_0^t$$

$$= \frac{10}{3} \lim_{t\to\infty} \tan^{-1}\left(\frac{t}{3}\right) = \frac{10}{3} \cdot \frac{\pi}{2} = \frac{5\pi}{3}.$$

Since the integral converges, the series also converges by the Integral Test.

6 We'll use the Limit Comparison Test, comparing the given series with $\sum_{n=1}^{\infty} \frac{1}{n}$. We have

$$\lim_{n\to\infty} \frac{\frac{1}{2n-\sqrt[3]{n^2}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{2n - n^{2/3}} = \lim_{n\to\infty} \frac{1}{2 - n^{-1/3}} = \frac{1}{2} \in (0, \infty),$$

and so since $\sum_{n=1}^{\infty} \frac{1}{n}$ is known to diverge, the given series must also diverge.

7 We have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left(\frac{[(n+1)!]^3}{[3(n+1)]!} \cdot \frac{(3n)!}{(n!)^3}\right) = \lim_{n\to\infty} \frac{(n+1)^3}{(3n+1)(3n+2)(3n+3)} = \frac{1}{27} \in [0, 1),$$

and so the series converges by the Ratio Test.

8 The Ratio Test will turn out to be inconclusive, so we use the Limit Comparison Test and compare with the divergent series $\sum_{n=1}^{\infty} \frac{1}{n}$, using L'Hôpital's Rule where indicated:

$$\lim_{n\to\infty} \frac{\ln\left(\frac{n+2}{n+1}\right)}{\frac{1}{n}} \overset{\text{LR}}{=} \lim_{n\to\infty} \frac{\frac{1}{n+2} - \frac{1}{n+1}}{-\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2 + 3n + 2} = 1 \in (0, \infty).$$

Thus the series diverges by the Limit Comparison Test.

9 We may write the series as

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}.$$

The Ratio Test will be inconclusive, so we try the Integral Test. With partial fraction decomposition we find that

$$\begin{aligned}\int_1^{\infty} \frac{1}{(2x-1)(2x+1)}\,dx &= \lim_{t\to\infty} \frac{1}{2} \int_1^t \left(\frac{1}{2x-1} - \frac{1}{2x+1}\right) dx \\ &= \frac{1}{4} \lim_{t\to\infty} \left[\ln\left(\frac{2x-1}{2x+1}\right)\right]_1^t = \frac{1}{4} \lim_{t\to\infty} \left[\ln\left(\frac{2t-1}{2t+1}\right) - \ln\tfrac{1}{3}\right] \\ &= \tfrac{1}{4}\left(\ln 1 - \ln\tfrac{1}{3}\right) = \tfrac{1}{4}\ln 3.\end{aligned}$$

Since the integral converges, we conclude that the series also converges.

10 For $n \geq 2$ we have

$$b_n = \frac{n-1}{4n^2+9} > 0,$$

with $b_n \to 0$ as $n \to \infty$. Is the sequence $(b_n)_{n=1}^{\infty}$ eventually nonincreasing, meaning $b_{n+1} \leq b_n$ for all sufficiently large n? We have

$$\begin{aligned} b_{n+1} \leq b_n \quad &\Leftrightarrow \quad \frac{n}{4(n+1)^2+9} \leq \frac{n-1}{9n^2+9} \quad \Leftrightarrow \quad n(4n^2+9) \leq (n-1)[4(n+1)^2+9] \\ &\Leftrightarrow \quad 0 \leq 4n^2 - 4n - 13 \quad \Leftrightarrow \quad 4n(n-1) \geq 13. \end{aligned}$$

Clearly $4n(n-1) \geq 13$ holds for all $n \geq 3$, and so $b_{n+1} \leq b_n$ holds for all $n \geq 3$. That is, $(b_n)_{n=1}^{\infty}$ is indeed eventually nonincreasing. Therefore, by the Alternating Series Test, we conclude that the series converges.