Math 141 Exam #1 Key (Fall 2016)

1 By definition the slope of the tangent line equals $(f^{-1})'(-3)$. We have $f(4) = -3$ and $f'(x) = 2x - 4$, so by the Inverse Function Theorem we have

$$(f^{-1})'(-3) = (f^{-1})'(f(4)) = \frac{1}{f'(4)} = \frac{1}{2(4)-4} = \frac{1}{4}.$$

2a Since $(\ln x)' = x^{-1}$, by the Product Rule we have

$$f'(x) = (x^{-1} \cdot \ln x)' = -x^{-2} \cdot \ln x + x^{-1} \cdot x^{-1} = -\frac{\ln x}{x^2} + \frac{1}{x^2} = \frac{1-\ln x}{x^2}.$$

2b We have

$$g'(x) = \frac{d}{dx} \exp\left[\ln\left(4+\tfrac{2}{x}\right)^{3x}\right] = \frac{d}{dx} \exp\left[3x\ln\left(4+\tfrac{2}{x}\right)\right] = \exp\left[3x\ln\left(4+\tfrac{2}{x}\right)\right] \frac{d}{dx}\left[3x\ln\left(4+\tfrac{2}{x}\right)\right]$$

$$= \left(4+\tfrac{2}{x}\right)^{3x}\left[3x \cdot \frac{1}{4+\frac{2}{x}} \cdot \left(-\tfrac{2}{x^2}\right) + 3\ln\left(4+\tfrac{2}{x}\right)\right] = \left(4+\frac{2}{x}\right)^{3x}\left[3\ln\left(4+\frac{2}{x}\right) - \frac{3}{2x+1}\right].$$

2c We have

$$h'(t) = \frac{d}{dt}\left[e^{\ln(\sin t)^{\sin t}}\right] = (\sin t)^{\sin t}\left[1+\ln(\sin t)\right]\cos t.$$

2d A formula could be used. Or let $y = r(x)$, so

$$y = \log_9\sqrt{3x} \quad\Rightarrow\quad 9^y = \sqrt{3x} \quad\Rightarrow\quad y\ln 9 = \frac{1}{2}\ln(3x) \quad\Rightarrow\quad y = \frac{\ln(3x)}{2\ln 9},$$

and hence

$$r'(x) = y' = \frac{d}{dx}\left[\frac{\ln(3x)}{2\ln 9}\right] = \frac{1}{2x\ln 9}.$$

2e In general $(\cos^{-1} x)' = -(\sin^{-1} x)'$, where the formula for $(\sin^{-1} x)'$ is given. Thus:

$$\varphi'(z) = -\frac{1}{z\sqrt{1-\ln^2 z}}.$$

Aufpassen: by definition $\ln^2 z = (\ln z)^2$.

2f $y' = 4\operatorname{sech}^3(\ln x) \cdot [-\tanh(\ln x)\operatorname{sech}(\ln x)] \cdot \dfrac{1}{x} = -\dfrac{4\operatorname{sech}^4(\ln x)\tanh(\ln x)}{x}.$

3 Note $(x^2)^x = x^{2x}$. Let $f(x) = x^{2x}$, which has domain $(0,\infty)$, and find $x \in (0,\infty)$ for which $f'(x) = 0$. That is, find $x > 0$ for which

$$(2 + 2\ln x)x^{2x} = 0.$$

This leads to $2 + 2\ln x = 0$, giving $\ln x = -1$, and finally $x = e^{-1}$. Therefore $y = (x^2)^x$ has a horizontal tangent line at the point $(e^{-1}, (e^{-2})^{e^{-1}})$.

4a Let $u = 4e^x + 6$, so $\frac{1}{4}du = e^x\,dx$, and we get

$$\int \frac{e^x}{4e^x+6}dx = \int \frac{1/4}{u}du = \frac{1}{4}\ln|u| + c = \frac{1}{4}\ln(4e^x+6) + c.$$

4b We have

$$\int \left(\frac{3}{p-6} - \frac{4}{8p+1}\right) dp = 3\ln|p-6| - \frac{1}{2}\ln|8p+1| + c.$$

4c Let $u = x^8$, so by the Substitution Rule we replace $x^7\,dx$ with $\frac{1}{8}du$ to get

$$\int x^7 8^{x^8}dx = \frac{1}{8}\int 8^u\,du = \frac{1}{8}\cdot\frac{8^u}{\ln 8} + c = \frac{8^{x^8}}{8\ln 8} + c.$$

4d Letting $u = \cosh t$, and noting that $\cosh t > 0$ for all $t \in \mathbb{R}$, we have

$$\int \frac{\sinh t}{1+\cosh t}dt = \int \frac{1}{1+u}du = \ln|u+1| + c = \ln|\cosh t + 1| + c = \ln(\cosh t + 1) + c.$$

5a Let $u = x^x$. Now, since $x^x = e^{\ln x^x} = e^{x\ln x}$, we have

$$\frac{du}{dx} = e^{x\ln x}(\ln x + 1) = (1 + \ln x)x^x.$$

Thus we formally have $du = (1+\ln x)x^x\,dx$ when we apply the Substitution Method, giving

$$\int_1^2 (1+\ln x)x^x\,dx = \int_1^4 du = 3.$$

5b Let $u = y^2$, so $y\,dy = \frac{1}{2}du$:

$$\int_1^{\sqrt{2}} y2^{y^2}dy = \frac{1}{2}\int_1^2 2^u du = \frac{1}{2}\left[\frac{2^u}{\ln 2}\right]_1^2 = \frac{1}{\ln 2}.$$

6 With LR indicating use of L'Hôpital's Rule, we have

$$\lim_{x\to 1^+} x^{1/(2-2x)} = \lim_{x\to 1^+}\exp\left(\frac{\ln x}{2-2x}\right) = \exp\left(\lim_{x\to 1^+}\frac{\ln x}{2-2x}\right) \overset{\text{LR}}{=} \exp\left(\lim_{x\to 1^+}\frac{1/x}{-2}\right) = e^{-1/2}.$$