Math 141 Final Exam Key (Fall 2015)

1a $f'(x) = \dfrac{1}{x + \ln x} \cdot (1 + 1/x) = \dfrac{x+1}{x^2 + x\ln x}$.

1b $g'(t) = e^{3t^2} + 6t^2 e^{3t^2}$.

1c $y' = \dfrac{4x^3}{\sqrt{1 - x^8}}$.

2a Substitution: let $u = \sqrt{x}$, so

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}} \quad \Rightarrow \quad 2\,du = \frac{1}{\sqrt{x}}\,dx,$$

and therefore

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = \int 2e^u du = 2e^u + C = 2e^{\sqrt{x}} + C.$$

2b We make the substitution $u = x/\sqrt{3}$:

$$\int \frac{4}{x^2+3}\,dx = \int \frac{4/3}{(x/\sqrt{3})^2 + 1}\,dx = \frac{4}{3}\int \frac{\sqrt{3}}{u^2+1}\,du = \frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C.$$

2c Use Integration by Parts. Letting $u(x) = \ln x$ and $v'(x) = x^2$, we obtain $u'(x) = 1/x$ and $v(x) = \frac{1}{3}x^3$, and so

$$\begin{aligned}\int_1^{e^2} x^2 \ln x\,dx &= \left[\frac{x^3}{3}\cdot \ln x\right]_1^{e^2} - \int_1^{e^2} \frac{x^3}{3}\cdot\frac{1}{x}\,dx = \frac{(e^2)^3 \ln e^2}{3} - \frac{1}{3}\int_1^{e^2} x^2\,dx \\ &= \frac{2e^6}{3} - \frac{1}{3}\left[\frac{1}{3}x^3\right]_1^{e^2} = \frac{2e^6}{3} - \frac{1}{9}(e^6 - 1) = \frac{5e^6+1}{9}.\end{aligned}$$

2d Letting $u = \cos x$,

$$\begin{aligned}\int \frac{\sin^5 x}{\cos^2 x}\,dx &= \int \frac{(1-\cos^2 x)^2 \sin x}{\cos^2 x}\,dx = -\int \frac{(1-u^2)^2}{u^2}\,du \\ &= \int (2 - u^{-2} - u^2)\,du = 2u + u^{-1} - \frac{1}{3}u^3 + C \\ &= 2\cos x + \sec x - \frac{\cos^3 x}{3} + C.\end{aligned}$$

2e Let $x = \frac{1}{2}\tan\theta$ (where we assume $-\pi/2 < \theta < \pi/2$), so $dx = \frac{1}{2}\sec^2\theta\, d\theta$, and then

$$\begin{aligned}\int \frac{1}{(1+\tan^2\theta)^{3/2}} \cdot \frac{1}{2}\sec^2\theta\, d\theta &= \frac{1}{2}\int \frac{\sec^2\theta}{(\sec^2\theta)^{3/2}}\, d\theta = \frac{1}{2}\int \frac{\sec^2\theta}{\sec^3\theta}\, d\theta \\ &= \frac{1}{2}\int \cos\theta\, d\theta = \frac{1}{2}\sin\theta + C = \frac{x}{\sqrt{1+4x^2}} + C,\end{aligned}$$

where $\sqrt{\sec^2\theta} = \sec\theta$ since $\theta \in (-\pi/2, \pi/2)$ implies $\sec\theta > 0$.

2f Apply partial fraction decomposition:

$$\int \frac{2}{x^2 - x - 6}\, dx = \int \left(\frac{2/5}{x-3} - \frac{2/5}{x+2}\right) dx = \frac{2}{5}\ln|x-3| - \frac{2}{5}\ln|x+2| + C.$$

3 We have

$$\int_0^\infty e^{-5x} dx = \lim_{t\to\infty}\int_0^t e^{-5x} dx = \lim_{t\to\infty}\left[-\frac{1}{5}e^{-5x}\right]_0^t = \lim_{t\to\infty}\frac{1}{5}(1 - e^{-5t}) = \frac{1}{5},$$

so the integral converges.

4 Recalling that the exponential function is continuous, we have

$$\begin{aligned}\lim_{n\to\infty}(1+2/n)^n &= \lim_{n\to\infty}\exp\big[\ln(1+2/n)^n\big] = \exp\Big[\lim_{n\to\infty}\ln(1+2/n)^n\Big] = \exp\left[\lim_{n\to\infty}\frac{\ln(1+2/n)}{1/n}\right] \\ &= \exp\left[\lim_{n\to\infty}\frac{(1+2/n)^{-1}\cdot(-2/n^2)}{-1/n^2}\right] = \exp\left[\lim_{n\to\infty}\frac{2}{1+2/n}\right] = \exp(2) = e^2,\end{aligned}$$

using L'Hôpital's Rule en route.

5 This is a convergent geometric series:

$$\sum_{k=1}^\infty \frac{5}{3^k} = \sum_{k=0}^\infty \frac{5}{3^{k+1}} = \frac{5}{3}\sum_{k=0}^\infty \left(\frac{1}{3}\right)^k = \frac{5}{3}\cdot\frac{1}{1-1/3} = \frac{5}{2}.$$

6a Since

$$\lim_{k\to\infty}\frac{k}{100k+3} = \lim_{k\to\infty}\frac{1}{100+3/k} = \frac{1}{100} \neq 0,$$

the series diverges by the Divergence Test.

6b Since

$$\lim_{k\to\infty} \frac{k}{\sqrt{k^2+4}} = \lim_{k\to\infty} \frac{1}{\sqrt{1+4/k^2}} = \frac{1}{\sqrt{1+0}} = 1 \neq 0,$$

the series diverges by the Divergence Test.

6c Since

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left[\frac{(k+1)^6}{(k+1)!} \cdot \frac{k!}{k^6}\right] = \lim_{k\to\infty} \left[\frac{1}{k+1}\left(\frac{k+1}{k}\right)^6\right]$$
$$= \lim_{k\to\infty} \left(\frac{1}{k+1}\right)\left[\lim_{k\to\infty}\left(1+\frac{1}{k}\right)\right]^6 = 0 \cdot 1^6 = 0 < 1,$$

the series converges by the Ratio Test.

6d Since

$$\lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} \sqrt[k]{k^2/2^k} = \lim_{k\to\infty} \frac{k^{2/k}}{2} = \frac{1}{2} < 1,$$

the Root Test concludes that the series converges.

6f Since $1/\sqrt{k^2+4}$ is monotone decreasing, with $1/\sqrt{k^2+4} \to 0$ as $k \to \infty$, the series converges by the Alternating Series Test.

7 We have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x/8)^{3(n+1)}}{(x/8)^{3n}}\right| = \lim_{n\to\infty} \left|\frac{x}{8}\right|^3.$$

By the Ratio Test the series converges if $|x/8|^3 < 1$, or equivalently $-8 < x < 8$. If $x = 8$, the series becomes $\sum_{n=0}^{\infty}(1)$, which diverges by the Divergence Test. If $x = -8$, the series becomes $\sum_{n=0}^{\infty}(-1)^{3n}$, which also diverges by the Divergence Test. Therefore $(-8, 8)$ is the interval of convergence of the series.

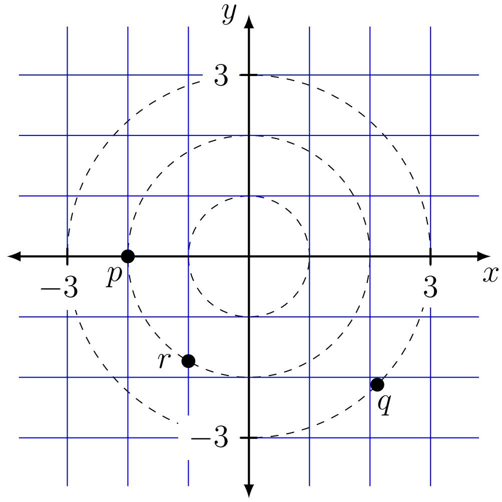
8 We have

$$e^{x/3} = 1 + \frac{x/3}{1!} + \frac{(x/3)^2}{2!} + \frac{(x/3)^3}{3!} + \cdots = 1 + \frac{x}{3} + \frac{x^2}{18} + \frac{x^3}{162} + \cdots$$

9 There are many possible parametrizations. One example:

$$x = 6\cos t, \quad y = 6\sin t, \quad t \in [0, 2\pi].$$

10 Letting $p = (2, \pi)$, $q = (3, -\pi/4)$, and $r = (-2, \pi/3)$:



11 Polar coordinates: $(r, \theta) = (4, \pi/3)$.

12 Since $x = r\cos\theta$ and $y = r\sin\theta$,

$$r = \cot\theta\csc\theta = \frac{\cos\theta}{\sin\theta}\cdot\frac{1}{\sin\theta} = \frac{\cos\theta}{\sin^2\theta} = \frac{r\cos\theta}{r\sin^2\theta} \quad\Rightarrow\quad r^2\sin^2\theta = r\cos\theta \quad\Rightarrow\quad y^2 = x.$$