Math 141 Exam #4 Key (Fall 2015)

1 Let $f(x) = \sqrt{x}$. The 3rd-order Taylor polynomial centered at 1 for f is

$$\begin{aligned} P_3(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f'''(1)}{6}(x-1)^3 \\ &= 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3, \end{aligned}$$

and so

$$\sqrt{1.06} = f(1.06) \approx P_3(1.06) = 1 + \frac{0.06}{2} - \frac{0.06^2}{8} + \frac{0.06^3}{16} = 1.0295635.$$

(Note this is very close to the actual value of 1.029563014.... Only a 0.0000472% error!)

2a Clearly the series converges when $x = 0$. Assuming $x \neq 0$, we find that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2 x^{2(n+1)}}{(n+1)!} \cdot \frac{n!}{n^2 x^{2n}}\right| = \lim_{n\to\infty}\frac{(n+1)x^2}{n^2} = 0$$

for all x, and so by the Ratio Test the series converges on $(-\infty, \infty)$. There are no endpoints to consider here.

2b Since

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{n+1} \cdot \frac{n}{(x-2)^n}\right| = \lim_{n\to\infty}\frac{n|x-2|}{n+1} = |x-2|,$$

by the Ratio Test the series converges if $|x-2| < 1$, which implies $x \in (1,3)$.

When $x = 1$ the series becomes

$$\sum \frac{(-1)^n}{n},$$

which converges by the Alternating Series Test. When $x = 3$ the series becomes

$$\sum \frac{1}{n},$$

which is the harmonic series and is known to diverge. The interval of convergence is therefore $[1,3)$.

2c Since

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}(4x-8)^{n+1}}{(n+1)} \cdot \frac{n}{2^n(4x-8)^n}\right| = \lim_{n\to\infty}\frac{2n|4x-8|}{n+1} = 2|4x-8|,$$

by the Ratio Test the series converges if $2|4x-8| < 1$, which implies $x \in \left(\frac{15}{8}, \frac{17}{8}\right)$.

When $x = 15/8$ the series becomes

$$\sum \frac{2^n}{n}\left(-\frac{1}{2}\right)^n = \sum \frac{(-1)^n}{n},$$

which converges by the Alternating Series Test.

When $x = 17/8$ the series becomes

$$\sum \frac{2^n}{n}\left(\frac{1}{2}\right)^n = \sum \frac{1}{n},$$

which is the harmonic series and is known to diverge. The interval of convergence is therefore $\left[\frac{15}{8}, \frac{17}{8}\right)$.

3 From the table provided we have

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

for $|x| < 1$. Hence

$$\sum_{n=0}^{\infty}\left(\frac{3}{2x^2+1}\right)^n = \frac{1}{1-\frac{3}{2x^2+1}} = \frac{2x^2+1}{2x^2-2}$$

for $3(2x^2+1)^{-1} < 1$. Now,

$$\frac{3}{2x^2+1} < 1 \quad\Rightarrow\quad 2x^2+1 > 3 \quad\Rightarrow\quad x^2 > 1 \quad\Rightarrow\quad x \in (-\infty,-1)\cup(1,\infty),$$

and so there are two intervals of convergence here: $(-\infty,-1)$ and $(1,\infty)$.

4a The first four nonzero terms of the Taylor series for f centered at -3 are

$$f(-3) + f'(-3)(x+3) + \frac{f''(-3)}{2}(x+3)^2 + \frac{f'''(-3)}{6}(x+3)^3.$$

Now,

$$f(x) = \frac{1}{x}, \quad f'(x) = -\frac{1}{x^2}, \quad f''(x) = \frac{2}{x^3}, \quad f'''(x) = -\frac{6}{x^4},$$

and so the first four nonzero terms are

$$-\frac{1}{3} - \frac{1}{9}(x+3) - \frac{1}{27}(x+3)^2 - \frac{1}{81}(x+3)^3.$$

4b Based on the pattern exhibited by the first five terms, we have

$$\sum_{n=0}^{\infty} \frac{-1}{3^{n+1}}(x+3)^n$$

5 From the table provided we have $e^x = \sum_{n=0}^{\infty} x^n/n!$ for all $x \in (-\infty,\infty)$, and so

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$$

for all x. Now,

$$\int\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^{2n}\right)dx=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}x^{2n+1}+c$$

for all x and arbitrary constant c. Thus, by the Fundamental Theorem of Calculus,

$$\int_0^{1/3}e^{-x^2}dx=\int_0^{1/3}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^{2n}\right)dx=\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}x^{2n+1}\right]_0^{1/3}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\frac{1}{3}\right)^{2n+1}.$$

We have arrived at an alternating series $\sum(-1)^n b_n$ with

$$b_n=\frac{1}{n!(2n+1)}\left(\frac{1}{3}\right)^{2n+1}$$

for $n\geq 0$. Evaluating the first few b_n values,

$$b_0=\tfrac{1}{3},\quad b_1=\tfrac{1}{81},\quad b_2=\tfrac{1}{2430},\quad b_3=\tfrac{1}{91{,}854},\quad b_4\approx 2.352\times 10^{-7},\quad b_5\approx 4.277\times 10^{-9},$$

and finally $b_6=6.701\times 10^{-11}$. By the Alternating Series Estimation Theorem the approximation

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\frac{1}{3}\right)^{2n+1}\approx b_0-b_1+b_2-b_3+b_4-b_5\approx 0.321388521$$

will have an absolute error less than $b_6\approx 6.701\times 10^{-11}<10^{-10}$; that is,

$$\int_0^{1/3}e^{-x^2}dx\approx b_0-b_1+b_2-b_3+b_4-b_5$$

has an absolute error less than 10^{-10}.

6 Here $x=\sqrt[5]{t}-2$ implies $t=(x+2)^5$, and so $y=t+1$ gives $y=(x+2)^5+1$. Thus we see that

$$f(x)=(x+2)^5+1,$$

and from $t\in[0,32]$ we see that $\mathrm{Dom}(f)=[-2,0]$.

7 There are many possible parametrizations, but one good one is

$$(x(t),y(t))=(8,2)(1-t)+(-2,-3)t=(8-10t,2-5t)$$

for $t\in[0,1]$.

8 It helps to multiply by r to get

$$r^2=2r\sin\theta+2r\cos\theta.$$

Then, since $x=r\cos\theta$, $y=r\sin\theta$, and $r^2=x^2+y^2$, we obtain

$$x^2+y^2=2y+2x.$$

We can improve on this: from $(x^2-2x)+(y^2-2y)=0$ we obtain

$$(x-1)^2+(y-1)^2=2,$$

which is seen to be the equation of a circle centered at $(1,1)$ with radius $\sqrt{2}$.

9 Using the identity $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$, area is

$$\begin{aligned}\mathcal{A} &= \frac{1}{2}\int_0^{2\pi}(2+\cos\theta)^2\,d\theta = \frac{1}{2}\int_0^{2\pi}(4+4\cos\theta+\cos^2\theta)\,d\theta \\ &= 2\int_0^{2\pi}d\theta + 2\int_0^{2\pi}\cos\theta\,d\theta + \frac{1}{4}\int_0^{2\pi}(1+\cos 2\theta)\,d\theta \\ &= 4\pi + 2\big[\sin\theta\big]_0^{2\pi} + \frac{1}{4}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{2\pi} = 4\pi + 0 + \frac{\pi}{2} = \frac{9\pi}{2}.\end{aligned}$$

And the exam is done.
And there is much rejoicing throughout the kingdom.