Math 141 Exam #3 Key (Fall 2015)

1a We have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2^n}{3^{n+1}} = \lim_{n\to\infty} \frac{1}{3}\left(\frac{2}{3}\right)^n = 0.$$

1b First we evaluate

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln\left(\frac{n+1}{2n}\right) = \lim_{n\to\infty} \ln\left(\frac{1+1/n}{2}\right) = \ln\left(\tfrac{1}{2}\right).$$

2 We have

$$\sum_{n=0}^{\infty} \frac{4^{n+1}}{5^n} = \sum_{n=0}^{\infty} 4\left(\frac{4}{5}\right)^n = \frac{4}{1-4/5} = 20.$$

3 For each $n \geq 1$ we have

$$\begin{aligned} s_k &= \sum_{n=1}^{k}\left(\frac{1}{n+6} - \frac{1}{n+7}\right) \\ &= \left(\frac{1}{7}-\frac{1}{8}\right)+\left(\frac{1}{8}-\frac{1}{9}\right)+\left(\frac{1}{9}-\frac{1}{10}\right)+\cdots+\left(\frac{1}{k+5}-\frac{1}{k+6}\right)+\left(\frac{1}{k+6}-\frac{1}{k+7}\right) \\ &= \frac{1}{7} - \frac{1}{k+7}, \end{aligned}$$

so

$$\sum_{n=1}^{\infty}\left(\frac{1}{n+6} - \frac{1}{n+7}\right) = \lim_{k\to\infty} s_k = \lim_{k\to\infty}\left(\frac{1}{7} - \frac{1}{k+7}\right) = \frac{1}{7}.$$

4a Since

$$\lim_{n\to\infty} \frac{n}{\sqrt{n^2+25}} = 1 \neq 0,$$

the series diverges by the Divergence Test.

4b Letting $u = -2x^2$, we have

$$\begin{aligned} \int_1^{\infty} xe^{-2x^2}\,dx &= \lim_{t\to\infty}\int_1^t xe^{-2x^2}\,dx = \lim_{t\to\infty}\int_{-2}^{-2t^2} -\frac{1}{4}e^u\,du = \lim_{t\to\infty} -\frac{1}{4}\left[e^u\right]_{-2}^{-2t^2} \\ &= \lim_{t\to\infty} -\frac{1}{4}\left(e^{-2t^2} - e^{-2}\right) = -\frac{1}{4}(0 - e^{-2}) = \frac{e^{-2}}{4}, \end{aligned}$$

so the integral converges, and therefore the series $\sum_{n=1}^{\infty} ne^{-2n^2}$ converges by the Integral Test.

4c Since

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}}{(n+1)^{99}}\cdot\frac{n^{99}}{2^n}\right| = \lim_{n\to\infty} 2\left(\frac{n}{n+1}\right)^{99} = 2\left(\lim_{n\to\infty}\frac{n}{n+1}\right)^{99} = 2(1)^{99} = 2 > 1,$$

the series diverges by the Ratio Test.

4d Since

$$\lim_{n\to\infty}|a_n|^{1/n} = \lim_{n\to\infty}\left[\left(\frac{n^2+1}{2n^2+1}\right)^n\right]^{1/n} = \lim_{n\to\infty}\frac{n^2+1}{2n^2+1} = \frac{1}{2} < 1,$$

the series converges by the Root Test.

4e For each $n \geq 1$ we have

$$0 \leq \frac{\sin^2 n}{n\sqrt{n}} \leq \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}},$$

and since $\sum_{n=1}^{\infty} n^{-3/2}$ is a convergent p-series, it follows that

$$\sum_{n=1}^{\infty}\frac{\sin^2 n}{n\sqrt{n}}$$

converges by the Direct Comparison Test.

4f For each $n \geq 1$ we have

$$0 \leq \frac{n^7}{n^9+3} \leq \frac{n^7}{n^9} = \frac{1}{n^2},$$

and since $\sum_{n=1}^{\infty} n^{-2}$ is a convergent p-series, it follows that

$$\sum_{n=1}^{\infty}\frac{n^7}{n^9+3}$$

converges by the Direct Comparison Test.

5a Since $\ln n$ and n are monotone increasing functions for $n \geq 2$, it follows that

$$\frac{1}{n\ln^2 n}$$

is monotone decreasing (i.e. nonincreasing) for $n \geq 2$. Also

$$\lim_{n\to\infty}\frac{1}{n\ln^2 n} = 0,$$

and so by the Alternating Series Test the series converges.

5b Since

$$\lim_{n\to\infty}\left|(-1)^n\left(1-\frac{2}{n}\right)\right| = \lim_{n\to\infty}\left(1-\frac{2}{n}\right) = 1 \neq 0,$$

the series diverges by the Divergence Test.