Math 141 Exam #2 Key (Fall 2015)

1 With completing the square we get $27-6\theta-\theta^2 = 36-(\theta+3)^2$. Now use a supplied formula to get

$$\int \frac{1}{\sqrt{27-6\theta-\theta^2}}d\theta = \int \frac{1}{\sqrt{36-(\theta+3)^2}}d\theta = \sin^{-1}\left(\frac{\theta+3}{6}\right)+c.$$

2a Use integration by parts twice:

$$\int e^x \sin x\,dx = -e^x\cos x + \int e^x \cos x\,dx = -e^x\cos x + \left(e^x \sin x - \int e^x \sin x\,dx\right),$$

which gives

$$2\int e^x \sin x\,dx = e^x\sin x - e^x \cos x,$$

and finally

$$\int e^x \sin x\,dx = \frac{e^x(\sin x - \cos x)}{2} + c.$$

2b The volume is given by

$$\mathcal{V} = \int_1^{e^2} \pi[f(x)]^2 dx = \pi\int_1^{e^2} x^2(\ln x)^2\,dx.$$

By integration by parts with $u(x) = (\ln x)^2$ and $v'(x) = x^2$ we have

$$\mathcal{V} = \pi\left(\left[\frac{x^3}{3}(\ln x)^2\right]_1^{e^2} - \frac{2}{3}\int_1^{e^2} x^2 \ln x\,dx\right) = \frac{4\pi e^6}{3} - \frac{2\pi}{3}\int_1^{e^2} x^2\ln x\,dx.$$

For the last integral again apply integration by parts, this time with $u(x) = \ln x$ and $v'(x) = x^2$, so

$$\int_1^{e^2} x^2 \ln x\,dx = \left[\frac{x^3}{3}\ln x\right]_1^{e^2} - \frac{1}{3}\int_1^{e^2} x^2\,dx = \frac{5e^6}{9} + \frac{1}{9}.$$

Therefore

$$\mathcal{V} = \frac{4\pi e^6}{3} - \frac{2\pi}{3}\left(\frac{5e^6}{9}+\frac{1}{9}\right) = \frac{2\pi(13e^6-1)}{27} \approx 1220.24.$$

(Note the *exact* answer is what is required here.)

3a We have

$$\int (\cos^3 x)\sqrt{\sin x}dx = \int (1-\sin^2 x)\sqrt{\sin x}\cos x\,dx,$$

so if we let $u = \sin x$, so that $\cos x\,dx$ is replaced by du by the Substitution Rule, we obtain

$$\int (\cos^3 x)\sqrt{\sin x}dx = \int (1-u^2)\sqrt{u}\,du = \int (u^{1/2}-u^{5/2})du$$
$$= \frac{2}{3}u^{3/2} - \frac{2}{7}u^{7/2} + c = \frac{2}{3}\sin^{3/2} x - \frac{2}{7}\sin^{7/2} x + c.$$

3b Let $u = \tan z$, so $du = \sec^2 z\, dz$ and we have

$$\int \frac{\sec^2 z}{\tan^5 z} dz = \int \frac{1}{u^5} du = -\frac{1}{4} u^{-4} + c = -\frac{1}{4\tan^4 z} + c.$$

3c Let $u = e^x + 1$, so $du = e^x\, dx$ and we have

$$\int e^x \sec(e^x + 1) dx = \int \sec u\, du = \ln|\sec u + \tan u| + c$$
$$= \ln\left|\sec(e^x + 1) + \tan(e^x + 1)\right| + c.$$

4a Let $x = \sin\theta$, so dx formally becomes $\cos\theta\, d\theta$. Now, $x \in \left[\frac{1}{2}, 1\right]$ implies $\frac{1}{2} \le \sin\theta \le 1$, and thus $\frac{\pi}{6} \le \theta \le \frac{\pi}{2}$. We have

$$\int_{1/2}^{1} \frac{\sqrt{1 - x^2}}{x^2} dx = \int_{\pi/6}^{\pi/2} \frac{\cos\theta}{\sin^2\theta} \cos\theta\, d\theta = \int_{\pi/6}^{\pi/2} (\csc^2\theta - 1)\, d\theta = -\big[\cot\theta + \theta\big]_{\pi/6}^{\pi/2}$$
$$= \left(\cot\frac{\pi}{6} + \frac{\pi}{6}\right) - \left(\cot\frac{\pi}{2} + \frac{\pi}{2}\right) = \sqrt{3} + \frac{\pi}{6} - 0 - \frac{\pi}{2} = \sqrt{3} - \frac{\pi}{3}.$$

4b Let $t = 13\sin\theta$ for $\theta \in [-\pi/2, \pi/2]$, so that dt is replaced with $13\cos\theta\, d\theta$ as part of the substitution. Observe that $-\pi/2 \le \theta \le \pi/2$ implies $\cos\theta \ge 0$, so that

$$\sqrt{\cos^2\theta} = |\cos\theta| = \cos\theta.$$

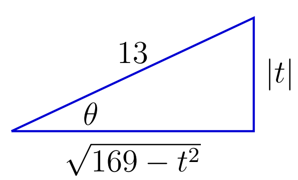
Now,

$$\int \sqrt{169 - t^2}\, dt = \int \sqrt{169 - 169\sin^2\theta} \cdot 13\cos\theta\, d\theta = \int 169\cos\theta\sqrt{1 - \sin^2\theta}\, d\theta$$
$$= 169 \int \cos\theta\sqrt{\cos^2\theta}\, d\theta = 169 \int \cos^2\theta\, d\theta,$$

and with the deft use of the given formula for $\int \cos^n\theta\, d\theta$ we obtain

$$\int \sqrt{169 - t^2}\, dt = 169\left(\frac{\cos\theta\sin\theta}{2} + \frac{1}{2}\int d\theta\right) = \frac{169}{2}\cos\theta\sin\theta + \frac{169}{2}\theta + c.$$

From $t = 13\sin\theta$ comes $\sin\theta = t/13$, so $\theta = \sin^{-1}(t/13)$ and θ may be characterized as an angle in the right triangle



Note that $t \geq 0$ if $\theta \in [0, \pi/2]$, and $t < 0$ if $\theta \in [-\pi/2, 0)$. From this triangle we see that $\cos\theta = \sqrt{169 - t^2}/13$, and therefore

$$\begin{aligned}\int \sqrt{169 - t^2}\, dt &= \frac{169}{2} \cdot \frac{\sqrt{169 - t^2}}{13} \cdot \frac{t}{13} + \frac{169}{2} \sin^{-1}\left(\frac{t}{13}\right) + c \\ &= \frac{t\sqrt{169 - t^2}}{2} + \frac{169}{2} \sin^{-1}\left(\frac{t}{13}\right) + c.\end{aligned}$$

5a We have

$$\frac{8}{(y-4)^2(y+3)} = \frac{A}{y-4} + \frac{B}{(y-4)^2} + \frac{C}{y+3},$$

so

$$8 = (A + C)y^2 + (-A + B - 8C)y + (-12A + 3B + 16C),$$

which yields the system of equations

$$\begin{cases} A \quad\quad + \quad C = 0 \\ -A + \ B - \ 8C = 0 \\ -12A + 3B + 16C = 8 \end{cases} \tag{1}$$

The solution to the system is $(A, B, C) = \left(-\frac{8}{49}, \frac{8}{7}, \frac{8}{49}\right)$, so

$$\begin{aligned}\int \frac{8}{(y-4)^2(y+3)} &= -\frac{8}{49}\int \frac{1}{y-4}dy + \frac{8}{7}\int \frac{1}{(y-4)^2}dy + \frac{8}{49}\int \frac{1}{y+3}dy \\ &= -\frac{8}{49}\ln|y-4| - \frac{8}{7(y-4)} + \frac{8}{49}\ln|y+3| + c \\ &= \frac{8}{49}\ln\left|\frac{y+3}{y-4}\right| - \frac{8}{7(y-4)} + c.\end{aligned}$$

5b Again start with a decomposition, noting that $x^2 + 2x + 6$ is an irreducible quadratic:

$$\begin{aligned}\int \frac{2}{(x-4)(x^2+2x+6)}\, dx &= \int \left(\frac{1/15}{x-4} + \frac{-x/15 - 2/5}{x^2+2x+6}\right) dx \\ &= \frac{1}{15}\ln|x-4| - \frac{1}{15}\int \frac{x+6}{(x+1)^2+5}\, dx.\end{aligned} \tag{2}$$

For the remaining integral, let $u = x + 1$ to obtain

$$\int \frac{x+6}{(x+1)^2+5}\, dx = \int \frac{u+5}{u^2+5}\, du = \int \frac{u}{u^2+5}\, du + 5\int \frac{1}{u^2 + (\sqrt{5}\,)^2}\, du \tag{3}$$

Letting $w = u^2 + 5$ in the first integral at right in (3), and using Formula (9) for the second, we next get

$$\int \frac{x+6}{(x+1)^2+5}\, dx = \int \frac{1/2}{w}\, dw + 5 \cdot \frac{1}{\sqrt{5}} \tan^{-1}\left(\frac{u}{\sqrt{5}}\right) + c$$

$$= \frac{1}{2}\ln|w| + \sqrt{5}\tan^{-1}\left(\frac{u}{\sqrt{5}}\right) + c = \frac{1}{2}\ln(u^2+5) + \sqrt{5}\tan^{-1}\left(\frac{u}{\sqrt{5}}\right) + c$$
$$= \frac{1}{2}\ln[(x+1)^2+5] + \sqrt{5}\tan^{-1}\left(\frac{x+1}{\sqrt{5}}\right) + c$$

Returning to (2),

$$\int \frac{2}{(x-4)(x^2+2x+6)}\,dx = \frac{\ln|x-4|}{15} - \frac{1}{15}\left[\frac{\ln[(x+1)^2+5]}{2} + \sqrt{5}\tan^{-1}\left(\frac{x+1}{\sqrt{5}}\right) + c\right]$$
$$= \frac{\ln|x-4|}{15} - \frac{\ln(x^2+2x+6)}{30} - \frac{\sqrt{5}}{15}\tan^{-1}\left(\frac{x+1}{\sqrt{5}}\right) + c.$$

6 The curve given by $y = (x+1)^{-3}$ for $x \in [0,\infty)$ is also given by $x = y^{-1/3} - 1$ for $y \in (0,1]$. The latter characterization allows us to determine that the volume in question is given by

$$\int_0^1 \pi(y^{-1/3}-1)^2\,dy = \pi \lim_{a\to 0^+}\int_a^1 (y^{-2/3} - 2y^{-1/3} + 1)\,dy = \pi \lim_{a\to 0^+}\left[3y^{1/3} - 3y^{2/3} + y\right]_a^1$$
$$= \pi \lim_{a\to 0^+}\left[1 - (3a^{1/3} - 3a^{2/3} + a)\right] = \pi.$$

7 First, we have

$$\int_0^1 \ln(y^2)\,dy = \lim_{a\to 0^+}\int_a^1 \ln(y^2)\,dy = 2\lim_{a\to 0^+}\int_a^1 \ln(y)\,dy = 2\lim_{a\to 0^+}\left[y\ln y - y\right]_a^1$$
$$= 2\lim_{a\to 0^+}\left[-1 - (a\ln a - a)\right] = -2.$$

By symmetry, then, we also have

$$\int_{-1}^0 \ln(y^2)\,dy = -2.$$

Therefore

$$\int_{-1}^1 \ln(y^2)\,dy = \int_{-1}^0 \ln(y^2)\,dy + \int_0^1 \ln(y^2)\,dy = -4.$$