Math 141 Exam #1 Key (Fall 2015)

1 We have $f'(x) = -2x$, so in particular $f'(1) = -2$. Now, the slope of the graph of f^{-1} at the point $(7,1)$ is $(f^{-1})'(7)$, where

$$(f^{-1})'(7) = \frac{1}{f'(1)} = -\frac{1}{2}$$

by the Inverse Function Theorem.

2a $f'(x) = -e^x \csc^2(e^x)$.

2b $g'(x) = \dfrac{1}{x \ln x}$.

2c Since $h(x) = e^{\ln(2x+1)^{2x}} = e^{2x\ln(2x+1)}$, we have

$$h'(x) = e^{2x\ln(2x+1)}[2x\ln(2x+1)]' = (2x+1)^{2x}\left[2\ln(2x+1) + \frac{4x}{2x+1}\right].$$

2d $x'(t) = \cos(\sec^{-1} 2t) \cdot \dfrac{2}{|2t|\sqrt{(2t)^2-1}} = \dfrac{\cos(\sec^{-1} 2t)}{|t|\sqrt{4t^2-1}}$.

2e $v'(y) = -\dfrac{1}{\sqrt{1-\left(\frac{1}{y^2+1}\right)^2}} \cdot \left(\frac{1}{y^2+1}\right)' = \dfrac{1}{\sqrt{1-\left(\frac{1}{y^2+1}\right)^2}} \cdot \dfrac{2y}{(y^2+1)^2} = \dfrac{2y}{(y^2+1)\sqrt{(y^2+1)^2-1}}$

2f $\varphi'(x) = 2x\cosh^2(3x) + 6x^2\cosh(3x)\sinh(3x)$.

3a $\displaystyle\int (2e^{-6x} + e^{9x})dx = -\frac{1}{3}e^{-6x} + \frac{1}{9}e^{9x} + c$

3b $\displaystyle\int \frac{9}{4-9y}dy = -\ln|4-9y| + c$

3c Let $u = x^8$, so by the Substitution Rule we replace $x^7\,dx$ with $\frac{1}{8}du$ to get

$$\int x^7 8^{x^8} dx = \frac{1}{8}\int 8^u\,du = \frac{1}{8}\cdot\frac{8^u}{\ln 8} + c = \frac{8^{x^8}}{8\ln 8} + c.$$

3d Letting $u = \cosh t$, and noting that $\cosh t > 0$ for all $t \in \mathbb{R}$, we have

$$\int \frac{\sinh t}{1+\cosh t} dt = \int \frac{1}{1+u} du = \ln|u+1| + c = \ln|\cosh t + 1| + c = \ln(\cosh t + 1) + c.$$

4a Let $u = x^x$. Now, since $x^x = e^{\ln x^x} = e^{x \ln x}$, we have

$$\frac{du}{dx} = e^{x \ln x}(\ln x + 1) = (1 + \ln x)x^x.$$

Thus we formally have $du = (1 + \ln x)x^x \, dx$ when we apply the Substitution Method, giving

$$\int_1^2 (1 + \ln x)x^x \, dx = \int_1^4 du = 3.$$

4b Let $u = e^x$, so formally $du = e^x \, dx$ and we have

$$\int_{-\ln\sqrt{3}}^0 \frac{e^x}{1+e^{2x}} dx = \int_{1/\sqrt{3}}^1 \frac{1}{1+u^2} du = \left[\tan^{-1}(u)\right]_{1/\sqrt{3}}^1 = \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}.$$

5 With LR indicating use of L'Hôpital's Rule, we have

$$\lim_{x\to1^+} \ln(x-1)^{\sin \pi x} = \lim_{x\to1^+} \frac{\ln(x-1)}{\csc \pi x} \overset{\text{LR}}{=} \lim_{x\to1^+} \frac{(x-1)^{-1}}{-\pi \csc \pi x \cot \pi x} = -\frac{1}{\pi} \lim_{x\to1^+} \frac{\sin \pi x \tan \pi x}{x-1}$$

$$\overset{\text{LR}}{=} -\frac{1}{\pi} \lim_{x\to1^+} \left(\pi \sin \pi x \sec^2 \pi x + \pi \cos \pi x \tan \pi x\right) = 0.$$

Thus

$$\lim_{x\to1^+} \left(\sqrt{x-1}\right)^{2\sin \pi x} = \lim_{x\to1^+} (x-1)^{\sin \pi x} = \lim_{x\to1^+} \exp\left[\ln(x-1)^{\sin \pi x}\right]$$

$$= \exp\left[\lim_{x\to1^+} \ln(x-1)^{\sin \pi x}\right] = \exp(0) = 1.$$