Math 141 Exam #4 Key (Fall 2011)

1a. Applying Ratio Test, $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(x+1)^{k+1}}{8^{k+1}}\cdot\frac{8^k}{(x+1)^k}\right| = \lim_{k\to\infty}\frac{|x+1|}{8} = \frac{|x+1|}{8}$, so series converges if $\frac{|x+1|}{8} < 1$, implying $-8 < x+1 < 8$ and thus $-9 < x < 7$. It remains to test the endpoints. When $x = 7$, $\lim_{k\to\infty}\left(\frac{x+1}{8}\right)^k = \lim_{k\to\infty}\left(\frac{7+1}{8}\right)^k = \lim_{k\to\infty}(1) = 1 \neq 0$, so the series diverges by the Divergence Test. When $x = -9$, $\lim_{k\to\infty}\left(\frac{x+1}{8}\right)^k = \lim_{k\to\infty}\left(\frac{-9+1}{8}\right)^k = \lim_{k\to\infty}(-1)^k \neq 0$, so again the series diverges. Therefore the interval of convergence is $(-9,7)$, and the radius of convergence is $|-9-7|/2 = 8$.

1b. Applying Ratio Test, $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(2x+3)^{k+1}}{6(k+1)}\cdot\frac{6k}{(2x+3)^k}\right| = \lim_{k\to\infty}\frac{k|2x+3|}{k+1} = |2x+3|$, so series converges if $-1 < 2x+3 < 1$, implying $-2 < x < -1$. When $x = -2$ series becomes $\sum_{k=1}^{\infty}\frac{(-1)^k}{6k}$, which converges by the Alternating Series Test. When $x = -1$ series becomes $\sum_{k=1}^{\infty}\frac{1}{6k}$, which diverges since $\sum_{k=1}^{\infty}\frac{1}{k}$ diverges. Interval of convergence is $[-2,-1)$, radius of convergence is $\frac{1}{2}$.

2. $g(x) = 5\cdot\frac{1}{1-6x} = \sum_{k=0}^{\infty} 5\cdot(6x)^k$, which converges if and only if $|6x| < 1$, so the interval of convergence is $(-\frac{1}{6},\frac{1}{6})$.

3. Use the geometric series given in the previous problem to get $f(x) = \frac{1}{1-(\sqrt{x}-7)} = \frac{1}{8-\sqrt{x}}$. The series converges if and only if $|\sqrt{x}-7| < 1$, which solves to give $6 < \sqrt{x} < 8$ and then $36 < x < 64$. So interval of convergence is $(36,64)$.

4a. $\frac{4^0}{0!}x^0 - \frac{4^2}{2!}x^2 + \frac{4^4}{4!}x^4 - \frac{4^6}{6!}x^6 + \cdots = 1 - \frac{16}{2}x^2 + \frac{256}{24}x^4 - \frac{4096}{720}x^6 + \cdots = 1 - 8x^2 + \frac{32}{3}x^4 - \frac{256}{45}x^6 + \cdots$

4b. $\sum_{k=0}^{\infty}\frac{(-1)^k 4^{2k}}{(2k)!}x^{2k} = \sum_{k=0}^{\infty}\frac{(-1)^k(4x)^{2k}}{(2k)!}$.

4c. Use the Ratio Test to find that the interval of convergence is $(-\infty,\infty)$.

5. Using the Maclaurin series for $\tan^{-1}(x)$ that is given, we obtain:

$$\begin{aligned}\lim_{x\to 0}\frac{3\tan^{-1}x - 3x + x^3}{x^5} &= \lim_{x\to 0}\frac{3\left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots\right) - 3x + x^3}{x^5} \\ &= \lim_{x\to 0}\frac{\frac{3}{5}x^5 - \frac{3}{7}x^7 + \cdots}{x^5} = \lim_{x\to 0}\left(\tfrac{3}{5} - \tfrac{3}{7}x^2 + \cdots\right) = \tfrac{3}{5}\end{aligned}$$

6. Use Ye Olde Remainder Theorem, which says: "Let $R_n = |S - S_n|$ be the remainder in approximating the value of a convergent alternating series $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ by the sum of its first n terms. Then $R_n \le a_{n+1}$." But before we can use this theorem there is some work to do. Because the Maclaurin series for $\sin(x)$ is everywhere convergent it can be multiplied by $1/x$ termwise:

$$\frac{\sin(x)}{x} = \frac{1}{x}\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k+1)!}.$$

Now, the new series we obtain is also everywhere convergent, so it can be integrated termwise:

$$\begin{aligned}\int_0^{0.15}\frac{\sin(x)}{x}\,dx &= \int_0^{0.15}\left[\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k+1)!}\right]dx = \sum_{k=0}^{\infty}\left[\int_0^{0.15}\frac{(-1)^k x^{2k}}{(2k+1)!}dx\right] = \sum_{k=0}^{\infty}\left[\frac{(-1)^k x^{2k+1}}{(2k+1)(2k+1)!}\right]_0^{0.15}\\ &= \sum_{k=0}^{\infty}\frac{(-1)^k(0.15)^{2k+1}}{(2k+1)(2k+1)!} = \frac{0.15}{(1)(1)} - \frac{0.15^3}{(3)(3!)} + \frac{0.15^5}{(5)(5!)} - \frac{0.15^7}{(7)(7!)} + \cdots\\ &\approx 0.15 - 1.875\times10^{-4} + 1.266\times10^{-7} - \cdots\end{aligned}$$

The Remainder Theorem assures us that if we estimate the value of $\int_0^{0.15}\sin(x)/x\,dx$ by $0.15 - 1.875\times10^{-4} \approx 0.1498$ then the error will be no greater than 1.266×10^{-7}, and this is certainly within our accepted tolerance of 10^{-4}! Therefore our estimate is 0.1498.

7. From $x = \sqrt{t} + 4$ comes $\sqrt{t} = x - 4$. Putting this into $y = 3\sqrt{t}$ gives $y = 3(x-4)$. Note that this will not be a line, since $0 \le t \le 16$ implies $0 \le \sqrt{t} \le 4$, and this means $0 \le x - 4 \le 4$. That is, we have $y = 3x - 12$ for $4 \le x \le 8$, which is a line segment.

8. $(4\sqrt{2}, \pi/4)$ and $(-4\sqrt{2}, 5\pi/4)$ will do, as well as $(4\sqrt{2}, -7\pi/4)$ and many other possibilities.

9. Multiply both sides by r to get $r^2 = 8r\sin\theta$. Now, since $r^2 = x^2 + y^2$ and $y = r\sin\theta$, we obtain $x^2 + y^2 = 8y$. Next we complete a square to see that we have a circle: $x^2 + (y^2 - 8y) = 0 \;\Rightarrow\; x^2 + (y^2 - 8y + 16) = 16 \;\Rightarrow\; x^2 + (y-4)^2 = 16$. The circle is centered at $(0,4)$ and has radius 4.

10. Set $f(\theta) = \sin 2\theta$. To find where horizontal tangent lines reside, find θ for which

$$\frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} = \frac{2\cos 2\theta\sin\theta + \sin 2\theta\cos\theta}{2\cos 2\theta\cos\theta - \sin 2\theta\sin\theta} = 0,$$

which entails solving $2\cos 2\theta\sin\theta + \sin 2\theta\cos\theta = 0$. Using the supplied identities $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta$, the equation becomes $(\sin\theta)(2 - 3\sin^2\theta) = 0$, so either $\sin\theta = 0$ or $\sin\theta = \pm\sqrt{2/3}$. Solving $\sin\theta = \sqrt{2/3}$ gives two solutions: $\theta_1 = \tan^{-1}\sqrt{2}$ (an angle in Quadrant I) and $\theta_2 = \pi - \tan^{-1}\sqrt{2}$ (in QII). Solving $\sin\theta = -\sqrt{2/3}$ gives $\theta_3 = \pi - \tan^{-1}(-\sqrt{2})$ (in QIII) and $\theta_4 = \tan^{-1}(-\sqrt{2})$ (in QIV). Putting these angles into $r = \sin 2\theta$ gives four points:

$$\left(\frac{2\sqrt{2}}{3}, \tan^{-1}\sqrt{2}\right),\ \left(-\frac{2\sqrt{2}}{3}, \pi - \tan^{-1}\sqrt{2}\right),\ \left(\frac{2\sqrt{2}}{3}, \pi - \tan^{-1}\left(-\sqrt{2}\right)\right),\ \left(\frac{-2\sqrt{2}}{3}, \tan^{-1}\left(-\sqrt{2}\right)\right).$$

(These types of problems are seldom pleasant company.) Moving on to $\sin\theta = 0$, we obtain $\theta = 0, \pi$, which yields just one point: $(0,0)$.