Math 141 Exam #3 Key (Fall 2011)

1a. $\displaystyle\lim_{n\to\infty}\frac{2n^{12}}{11n^{12}+4n^5}=\lim_{n\to\infty}\frac{2}{11+4/n^7}=\frac{2}{11+0}=\frac{2}{11}$

1b. First we evaluate $\displaystyle\lim_{n\to\infty}\sqrt[n]{n}=\lim_{n\to\infty}n^{1/n}=\lim_{n\to\infty}\exp(\ln n^{1/n})=\exp\left(\lim_{n\to\infty}\ln n^{1/n}\right)=\exp\left(\lim_{n\to\infty}\frac{\ln n}{n}\right)\overset{LR}{=}$ $\displaystyle\exp\left(\lim_{n\to\infty}\frac{1}{n}\right)=\exp(0)=1.$

Now, consider the subsequence of $\{a_n\}_{n=1}^{\infty}$ that consists of the even-indexed terms, which can be denoted by $\{a_{n_k}\}_{k=1}^{\infty}$ with $n_k=2k$ for $k\geq 1$. Then, using the fact that $\lim\limits_{n\to\infty}n^{1/n}=1$, we have $\lim\limits_{k\to\infty}a_{n_k}=\lim\limits_{k\to\infty}(-1)^{n_k}n_k^{1/n_k}=$ $\lim\limits_{k\to\infty}(-1)^{2k}(2k)^{1/(2k)}=\lim\limits_{k\to\infty}(2k)^{1/(2k)}=1.$

Next, consider the subsequence consisting of the odd-indexed terms, which can be denoted by $\{a_{n_k}\}_{k=1}^{\infty}$ with $n_k=2k-1$ for $k\geq 1$. Then we have $\lim\limits_{k\to\infty}a_{n_k}=\lim\limits_{k\to\infty}(-1)^{2k-1}(2k-1)^{1/(2k-1)}=\lim\limits_{k\to\infty}\left[-(2k-1)^{1/(2k-1)}\right]=-1.$

Since $\{a_n\}$ has two subsequences with different limits, the sequence $\{a_n\}$ itself cannot converge. That is, $\{a_n\}$ diverges.

1c. For all $n\geq 1$ we have $-1\leq\cos n\leq 1$, and thus $\displaystyle -\frac{1}{2^n}\leq\frac{\cos n}{2^n}\leq\frac{1}{2^n}$ for all n. Since $\displaystyle\lim_{n\to\infty}\frac{-1}{2^n}=0=\lim_{n\to\infty}\frac{1}{2^n}$, by the Squeeze Theorem we conclude that $\displaystyle\lim_{n\to\infty}\frac{\cos n}{2^n}=0.$

2. $\displaystyle\sum_{k=2}^{\infty}\frac{5}{3^k}=\sum_{k=0}^{\infty}\frac{5}{3^{k+2}}=\sum_{k=0}^{\infty}\frac{5}{9}\left(\frac{1}{3}\right)^k=\frac{5}{9}\cdot\frac{1}{1-1/3}=\frac{5}{9}\cdot\frac{3}{2}=\frac{5}{6}.$

3. Partial fraction decomposition gives $\displaystyle\frac{1}{(k+1)(k+2)}=\frac{1}{k+1}-\frac{1}{k+2}$, so series becomes $\displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{k+1}-\frac{1}{k+2}\right).$

Now, $\displaystyle s_n=\sum_{k=1}^{n}\left(\frac{1}{k+1}-\frac{1}{k+2}\right)=\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)+\left(\frac{1}{n+1}-\frac{1}{n+2}\right)=$ $\displaystyle\frac{1}{2}-\frac{1}{n+2}$, so $\displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{k+1}-\frac{1}{k+2}\right)=\lim_{n\to\infty}s_n=\lim_{n\to\infty}\left(\frac{1}{2}-\frac{1}{n+2}\right)=\frac{1}{2}.$

4. $1/k^k<10^{-3}$ is true for integers $k\geq 5$. By the Remainder Theorem, then, the absolute error will be less than 10^{-3} if we estimate the series by $\displaystyle\sum_{k=1}^{4}\frac{(-1)^k}{k^k}=-\frac{1}{1^1}+\frac{1}{2^2}-\frac{1}{3^3}+\frac{1}{4^4}\approx-0.7831.$

5a. $\displaystyle\lim_{k\to\infty}\frac{k}{99k+50}=\frac{1}{99}\neq 0$, so series diverges by the Divergence Test.

5b. $\displaystyle\lim_{k\to\infty}\frac{k}{\sqrt{k^2+4}}=\lim_{k\to\infty}\frac{1}{\sqrt{1+4/k^2}}=\frac{1}{\sqrt{1+0}}=1\neq 0$, so the series diverges by the Divergence Test.

5c. $\lim_{k\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{k\to\infty}\left|\frac{[(k+1)!]^2}{[2(k+1)]!}\cdot\frac{(2k)!}{(k!)^2}\right| = \lim_{k\to\infty}\frac{(k+1)(k+1)}{(2k+1)(2k+2)} = \lim_{k\to\infty}\frac{k+1}{4k+2} = \frac{1}{4} < 1$, so Ratio Test concludes that the series converges.

5d. $\lim_{k\to\infty}\sqrt[k]{|a_k|} = \lim_{k\to\infty}\sqrt[k]{k^2/2^k} = \lim_{k\to\infty}\frac{k^{2/k}}{2} = 1/2 < 1$, so Root Test concludes that the series converges.

5e. Use the Limit Comparison Test on the series $\sum_{k=2}^{\infty}\frac{k^2-1}{k^3+9}$ and $\sum_{k=2}^{\infty}\frac{1}{k}$, starting the index k at 2 since, technically, the test requires the series involved to consist of *positive* terms. It's known that $\sum_{k=1}^{\infty}1/k$ diverges, so therefore $\sum_{k=2}^{\infty}\frac{1}{k}$ diverges also. Now, since $\lim_{k\to\infty}\frac{\frac{k^2-1}{k^3+9}}{\frac{1}{k}} = \lim_{k\to\infty}\frac{k^3-k}{k^3+9} = 1 \in (0,\infty)$, the LCT concludes that $\sum_{k=2}^{\infty}\frac{k^2-1}{k^3+9}$ must diverge. Therefore the original series $\sum_{k=1}^{\infty}\frac{k^2-1}{k^3+9}$ diverges.

5f. For all $k \geq 1$ we have $\frac{k^8}{k^{11}+3} \leq \frac{k^8}{k^{11}} = \frac{1}{k^3}$, and since $\sum_{k=1}^{\infty}\frac{1}{k^3}$ is a convergent p-series, it follows by the Direct Comparison Test that the series $\sum_{k=1}^{\infty}\frac{k^8}{k^{11}+3}$ converges.

6a. Since $\ln k$ and k are monotone increasing functions for $k \geq 2$, it follows that $\frac{1}{k\ln^2 k}$ is monotone decreasing (i.e. nonincreasing) for $k \geq 2$. Also $\lim_{k\to\infty}\frac{1}{k\ln^2 k} = 0$, and so by the Alternating Series Test the series converges.

6b. Since $\lim_{k\to\infty}\left|(-1)^k\left(1+\frac{2}{k}\right)\right| = \lim_{k\to\infty}\left(1+\frac{2}{k}\right) = 1 \neq 0$, the series diverges by the Divergence Test.