MATH 141 EXAM #2 KEY (FALL 2011)

1. Letting $u = 4x$ and using a reduction formula, $\int \tan^3(4x)\,dx = \frac{1}{4}\int \tan^3 u\,du = \frac{1}{4}\left(\frac{\tan^2 u}{2} - \int \tan u\,du\right)$ $= \frac{1}{4}\left(\frac{\tan^2 u}{2} - \ln|\sec u| + C\right) = \frac{1}{8}\tan^2(4x) - \frac{1}{4}\ln|\sec(4x)| + C.$

2. Letting $u = \cos x$, $\int \frac{\sin^5 x}{\cos^2 x}\,dx = \int \frac{(1-\cos^2 x)^2 \sin x}{\cos^2 x}\,dx = -\int \frac{(1-u^2)^2}{u^2}\,du = \int (2 - u^{-2} - u^2)\,du =$ $2u + u^{-1} - \frac{1}{3}u^3 + C = 2\cos x + \sec x - \frac{\cos^3 x}{3} + C$

3. Let $x = 9\sec\theta$, so $dx = 9\sec\theta\tan\theta\,d\theta$ and we obtain $\int \frac{1}{\sqrt{81\sec^2\theta - 81}} \cdot 9\sec\theta\tan\theta\,d\theta = \int \frac{\sec\theta\tan\theta}{\sqrt{\tan^2\theta}}\,d\theta =$ $\int \frac{\sec\theta\tan\theta}{\tan\theta}\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{x}{9} + \frac{\sqrt{x^2-81}}{9}\right| + C$, where $\sqrt{\tan^2\theta} = \tan\theta$ since $x > 9 \Rightarrow \sec\theta > 1 \Rightarrow 0 < \theta < \pi/2$, so $\tan\theta > 0$.

4. Let $x = \frac{1}{2}\tan\theta$ (where we assume $-\pi/2 < \theta < \pi/2$), so $dx = \frac{1}{2}\sec^2\theta\,d\theta$, and $\int \frac{1}{(1+\tan^2\theta)^{3/2}} \cdot \frac{1}{2}\sec^2\theta\,d\theta =$ $\frac{1}{2}\int \frac{\sec^2\theta}{(\sec^2\theta)^{3/2}}\,d\theta = \frac{1}{2}\int \frac{\sec^2\theta}{\sec^3\theta}\,d\theta = \frac{1}{2}\int \cos\theta\,d\theta = \frac{1}{2}\sin\theta + C = \frac{x}{\sqrt{1+4x^2}} + C$, where $\sqrt{\sec^2\theta} = \sec\theta$ since $\theta \in (-\pi/2, \pi/2)$ implies $\sec\theta > 0$.

5. $\int \frac{2}{x^2 - x - 6}\,dx = \int \left(\frac{2/5}{x-3} - \frac{2/5}{x+2}\right)dx = \frac{2}{5}\ln|x-3| - \frac{2}{5}\ln|x+2| + C.$

6. $\int \frac{y}{(y-6)(y+2)^2}\,dy = \int \left(\frac{3/32}{y-6} - \frac{3/32}{y+2} + \frac{1/4}{(y+2)^2}\right)dy = \frac{3}{32}\ln|y-6| - \frac{3}{32}\ln|y+2| - \frac{1}{4(y+2)} + C.$

7. $\int \frac{z+1}{z(z^2+4)}\,dz = \int \left(\frac{1/4}{z} - \frac{z/4 - 1}{z^2+4}\right)dz = \frac{1}{4}\ln|z| - \frac{1}{8}\ln(z^2+4) + \frac{1}{2}\tan^{-1}\left(\frac{z}{2}\right) + C.$

8. $\int_0^\infty e^{-5x}dx = \lim_{b\to\infty}\int_0^b e^{-5x}dx = \lim_{b\to\infty}\left[-\frac{1}{5}e^{-5x}\right]_0^b = \lim_{b\to\infty}\frac{1}{5}(1 - e^{-5b}) = \frac{1}{5}$, so the integral converges.

9. Letting $u = x^4 - 1$, we obtain $\int_0^1 \frac{x^3}{x^4-1}\,dx = \lim_{c\to 1^-}\int_0^c \frac{x^3}{x^4-1}\,dx = \lim_{c\to 1^-}\int_{-1}^{c^4-1}\frac{1}{4u}\,du = \lim_{c\to 1^-}\left[\frac{1}{4}\ln|u|\right]_{-1}^{c^4-1} =$ $\lim_{c\to 1^-}\frac{1}{4}\ln|c^4 - 1| = -\infty$. Hence the integral diverges.

10a. $1/3^5 = 1/243$ and $1/3^6 = 1/729$

10b. $a_n = \frac{1}{3}a_{n-1}$, $a_0 = 1$

10c. $a_n = 1/3^n$, $n \geq 0$