Math 141 Exam #1 Key (Fall 2011)

1. Set $y = 5x - 9$ and solve for x to get $x = (y+9)/5$. Hence $f^{-1}(x) = (x+9)/5$.

2. Set $y = 3/(x^2+6)$ and solve for x to get $x^2 + 6 = 3/y$ and thus $x = \pm\sqrt{3/y - 6}$. If we assume $x \geq 0$ then we get $x = \sqrt{3/y-6}$, and if we assume $x \leq 0$ then we get $x = -\sqrt{3/y-6}$. Thus one inverse is $f_1^{-1}(x) = \sqrt{3/x-6}$ (the inverse of f restricted to $[0,\infty)$), with $\mathrm{Dom}(f_1^{-1}) = (0, 1/2]$; and another inverse is $f_2^{-1}(x) = -\sqrt{3/x-6}$ (the inverse of f restricted to $(-\infty, 0]$), with $\mathrm{Dom}(f_2^{-1}) = (0,1/2]$ also.

3. We're not able to get f^{-1} directly, so we must employ the theorem as follows: "If f is one-to-one and differentiable on an open interval I, $a \in I$, and $f(a) = b$, then $(f^{-1})'(b) = 1/f'(a)$ if $f'(a) \neq 0$." First we must find a for which $f(a) = 3$, which requires solving $a^3 + a + 1 = 3$. This is a knotty equation to solve analytically but one obvious solution is $a = 1$, and actually this is the *only* real solution since f (which is differentiable everywhere) is seen to be one-to-one by examining its derivative: $f'(x) = 3x^2 + 1 > 0$ for all $x \in \mathbb{R}$, so f must be strictly increasing on $\mathbb{R}$. Now, since $f(1) = 3$, we have $(f^{-1})'(3) = 1/f'(1) = 1/4$.

4. Let $f(x) = \ln(\ln(x))$. Then $f'(x) = [\ln(\ln(x))]' = \ln'(\ln(x)) \cdot \ln'(x) = \dfrac{1}{\ln(x)} \cdot \dfrac{1}{x} = \dfrac{1}{x\ln(x)}$. The domain for f' must first of all be a subset of $\mathrm{Dom}(f)$, where $\mathrm{Dom}(f) = \{x : x \in \mathrm{Dom}(\ln) \text{ and } \ln(x) \in \mathrm{Dom}(\ln)\}$. Working this out yields $\mathrm{Dom}(f) = \{x : x > 0 \text{ and } \ln(x) > 0\} = \{x : x > 0 \text{ and } x > 1\} = (1,\infty)$. Looking at the expression for $f'(x)$ that we derived, we see that no other difficulties arise for any $x > 1$. So $\mathrm{Dom}(f') = (1,\infty)$ also.

5. $f'(x) = e^{\sin(2x)} \cdot [\sin(2x)]' = e^{\sin(2x)} \cdot 2\cos(2x) = 2\cos(2x)e^{\sin(2x)}$, and so $f'(\pi/4) = 2\cos(\pi/2)e^{\sin(\pi/2)} = 0$.

6a. Let $u = x+1$, so $x = u - 1$ and $du = dx$, giving $\displaystyle\int_1^4 \frac{2(u-1)-1}{u}\,du = \int_1^4 \left(2 - \frac{3}{u}\right) du = [2u - 3\ln|u|]_1^4 =$ $[2(4) - 3\ln 4] - [2(1) - 3\ln 1] = 6 - 3\ln 4$.

6b. Let $u = e^x - e^{-x}$, so $du = (e^x + e^{-x})dx$ and we obtain $\displaystyle\int \frac{e^x + e^{-x}}{e^x - e^{-x}}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln|e^x - e^{-x}| + C$.

6c. $\displaystyle\int_{-2}^2 4^x\,dx = \int_{-2}^2 e^{\ln 4^x}\,dx = \int_{-2}^2 e^{x\ln 4}\,dx = \left[\frac{1}{\ln 4}e^{x\ln 4}\right]_{-2}^2 = \frac{1}{\ln 4}\left(e^{2\ln 4} - e^{-2\ln 4}\right) = \frac{1}{\ln 4}(16 - 1/16) = \frac{255}{16\ln 4}$.

6d. $\displaystyle\int \frac{5}{\sqrt{7^2 - x^2}}\,dx = 5\sin^{-1}\left(\frac{x}{7}\right) + C$.

7. We have $\ln[f(x)] = \ln\left[(\tan x)^{\sin x}\right] = (\sin x)\ln(\tan x)$, and so, differentiating both sides, we obtain $\dfrac{f'(x)}{f(x)} =$ $(\cos x)\ln(\tan x) + (\sin x) \cdot \dfrac{\sec^2 x}{\tan x} = (\cos x)\ln(\tan x) + \sec x$. Thus $f'(x) = f(x)[(\cos x)\ln(\tan x) + \sec x]$, and finally $f'(x) = (\tan x)^{\sin x}[(\cos x)\ln(\tan x) + \sec x]$.

8a. $s'(t) = -\sin(2^t) \cdot (2^t)' = -\sin(2^t) \cdot 2^t \cdot \ln 2 = -(2^t \ln 2)\sin(2^t)$.

8b. $f'(x) = 4 \cdot \dfrac{1}{(x^2-1)\ln 3} \cdot (x^2 - 1)' = \dfrac{8x}{(x^2-1)\ln 3}$

8c. $g'(y) = -\sin(\sin^{-1}(2y)) \cdot (\sin^{-1}(2y))' = -\sin(\sin^{-1}(2y)) \cdot \dfrac{1}{\sqrt{1-(2y)^2}} \cdot (2y)' = -\dfrac{4y}{\sqrt{1-4y^2}}$.

8d. $h'(z) = \dfrac{1}{|\ln z|\sqrt{(\ln z)^2 - 1}} \cdot (\ln z)' = \dfrac{1}{z|\ln z|\sqrt{\ln^2 z - 1}}$.

9. $\lim\limits_{x\to0^+}(1+x)^{\cot x} = \lim\limits_{x\to0^+}\exp\left[\ln(1+x)^{\cot x}\right] = \exp\left[\lim\limits_{x\to0^+}\ln(1+x)^{\cot x}\right] = \exp\left[\lim\limits_{x\to0^+}\dfrac{\ln(1+x)}{\tan x}\right]$. The limit is a $0/0$ indeterminate form, so we use L'Hôpital's Rule to get:

$$\lim_{x\to0^+}(1+x)^{\cot x} = \exp\left[\lim_{x\to0^+}\frac{1/(1+x)}{\sec^2 x}\right] = \exp\left(\lim_{x\to0^+}\frac{\cos^2 x}{1+x}\right) = \exp\left(\frac{\cos^2 0}{1+0}\right) = \exp(1) = e.$$

10a. In the integration by parts formula, let $u(x) = x^2$ and $v'(x) = e^{4x}$, so that $u'(x) = 2x$ and $v(x) = \frac{1}{4}e^{4x}$. We obtain: $\displaystyle\int x^2 e^{4x}\,dx = \frac{x^2}{4}e^{4x} - \int 2x \cdot \frac{1}{4}e^{4x}\,dx = \frac{x^2}{4}e^{4x} - \frac{1}{2}\int xe^{4x}\,dx$. To resolve the last integral we employ integration by parts a second time: letting $u(x) = x$ and $v'(x) = e^{4x}$, so that $u'(x) = 1$ and $v(x) = \frac{1}{4}e^{4x}$, we obtain: $\displaystyle\int xe^{4x}\,dx = \frac{x}{4}e^{4x} - \int \frac{1}{4}e^{4x}\,dx = \frac{x}{4}e^{4x} - \frac{1}{16}e^{4x} + C$. Finally, at long last, we find that $\displaystyle\int x^2e^{4x}\,dx = \frac{x^2}{4}e^{4x} - \frac{1}{2}\left(\frac{x}{4}e^{4x} - \frac{1}{16}e^{4x} + C\right) = \left(\frac{x^2}{4} - \frac{x}{8} + \frac{1}{32}\right)e^{4x} + C = \frac{e^{4x}}{32}(8x^2 - 4x + 1) + C$.

10b. Let $u(x) = x$ and $v'(x) = \cos 2x$, so that $u'(x) = 1$ and $v(x) = \frac{1}{2}\sin 2x$, and integration by parts yields: $\displaystyle\int_0^{\pi/2} x\cos 2x\,dx = \frac{1}{2}\left[x\sin 2x\right]_0^{\pi/2} - \frac{1}{2}\int_0^{\pi/2}\sin 2x\,dx = -\frac{1}{2}\left[-\frac{1}{2}\cos 2x\right]_0^{\pi/2} = \frac{1}{4}(\cos\pi - \cos 0) = -\frac{1}{2}$