MATH 140 EXAM #2 KEY (SUMMER II 2012)

1. Using the general result $\lim_{x\to 0}(\sin ax)/x = a$, we have

$$\lim_{x\to 0}\frac{\tan 9x}{\sin x} = \lim_{x\to 0}\frac{\sin 9x}{\cos 9x}\cdot\frac{1}{\sin x} = \lim_{x\to 0}\left(\frac{\sin 9x}{x}\cdot\frac{x}{\sin x}\cdot\frac{1}{\cos 9x}\right)$$

$$= \lim_{x\to 0}\left(\frac{\sin 9x}{x}\right)\cdot\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{-1}\cdot\lim_{x\to 0}\left(\frac{1}{\cos 9x}\right) = (9)(1)^{-1}(1) = 9.$$

2. Quotient Rule:

$$f'(x) = \frac{(2-\tan x)(\cos x)' - (\cos x)(2-\tan x)'}{(2-\tan x)^2}$$

$$= \frac{(2-\tan x)(-\sin x) - (\cos x)(-\sec^2 x)}{(2-\tan x)^2} = \frac{\tan x\sin x - 2\sin x + \sec x}{(2-\tan x)^2}$$

3. From $y'(x) = 8(\cos^2 x - \sin^2 x)$ we have

$$y'(\pi/3) = 8\cos^2(\pi/3) - 8\sin^2(\pi/3) = 8(1/2)^2 - 8(\sqrt{3}/2)^2 = 2 - 8(3/4) = -4$$

as the slope of the tangent line. The equation is thus

$$y - 8\cos(\pi/3)\sin(\pi/3) = -4(x - \pi/3),$$

or

$$y = -4x + 4\pi/3 + 2\sqrt{3}.$$

4a. $f'(x) = 13(5x^3 - x)^{12}(15x^2 - 1).$

4b. $g'(t) = \sin(4\cot t)\cdot(4\cot t)' = \sin(4\cot t)\cdot(-4\csc^2 t) = -4\csc^2 t\sin(4\cot t).$

4c. $h'(x) = \frac{1}{2}(x + x^{1/2})^{-1/2}\cdot(x + x^{1/2})' = \frac{1}{2}(x + x^{1/2})^{-1/2}\cdot(1 + \frac{1}{2}x^{-1/2})$, or

$$h'(x) = \frac{1 + \frac{1}{2}x^{-1/2}}{2\sqrt{x + \sqrt{x}}} = \frac{2\sqrt{x} + 1}{4\sqrt{x}\sqrt{x + \sqrt{x}}} = \frac{2\sqrt{x} + 1}{4\sqrt{x^2 + x\sqrt{x}}}.$$

5. From $[\cos(y^2) + 2x]' = (y^3)'$ we get

$$-\sin(y^2)\cdot 2yy' + 2 = 3y^2y' \quad\Rightarrow\quad 2yy'\sin(y^2) + 3y^2y' = 2 \quad\Rightarrow\quad y' = \frac{2}{2y\sin(y^2) + 3y^2}.$$

6. From $(x^4)' = (2x^2 + 2y^2)'$ we get

$$4x^3 = 4x + 4yy' \quad \Rightarrow \quad y' = \frac{4x^3 - 4x}{4y} \quad \Rightarrow \quad y' = \frac{x^3 - x}{y}.$$

Slope of tangent line is thus $y' = (2^3 - 2)/2 = 3$, so equation is $y - 2 = 3(x - 2)$, or $y = 3x - 4$.

7. Area of rectangle at time t is $A(t) = (2 + t)(4 + t) = t^2 + 6t + 8$. Rate of change of the area at time t is $A'(t) = 2t + 6$. Thus at time $t = 20$ seconds the area is increasing at a rate of $A'(20) = 2(20) + 6 = 46$ cm^2/s.

8. Let x be the distance between the man and the base of the street light, and let ℓ be the length of the man's shadow. The triangles $\triangle ABC$ and $\triangle OBD$ in the figure below are similar, and so we have

$$\frac{\ell + x}{7} = \frac{\ell}{2}.$$

Solving this for ℓ and observing that ℓ and x are both functions of time t, we obtain

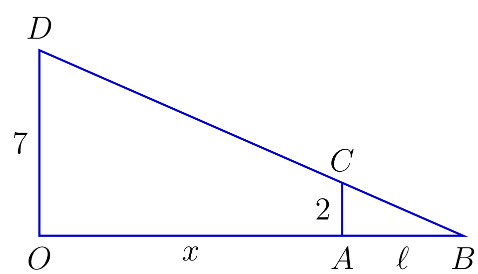
$$\ell(t) = \frac{2}{5}x(t).$$

Differentiating both sides with respect to t gives

$$\ell'(t) = \frac{2}{5}x'(t) = \frac{2}{5} \cdot (-1) = -\frac{2}{5} \text{ m/s}.$$

Thus at any time t the length of the shadow is growing shorter at a rate of $\frac{2}{5}$ m/s, which includes the time when the man is 5 meters from the street light!

Regarding the rate at which the tip of the shadow is moving, since point B in the figure—which is the tip of the shadow—is moving toward A at $-\frac{2}{5}$ m/s, and A is moving toward O at -1 m/s, is follows that B is moving toward O at $-1\frac{2}{5}$ m/s.



9. We have $f'(x) = 3x^2 - 4x - 5$. Setting $f'(x) = 0$ gives the quadratic equation $3x^2 - 4x - 5 = 0$, which has solutions

$$x = \frac{2 \pm \sqrt{19}}{3} \approx 2.12, \; -0.79.$$

Neither of these critical points lies in $[4, 8]$, so we need only evaluate f at the endpoints of the interval: $f(4) = 18$ is the global minimum and $f(8) = 350$ the global maximum.

10a. $\text{Dom}(f) = (-\infty, \infty)$.

10b. Since $f(0) = 0$, the y-intercept of f, which doubles as an x-intercept, is $(0,0)$. As for any x-intercepts besides the origin, we set $f(x) = 0$ and solve for x:

$$f(x) = 0 \quad \Rightarrow \quad x^{1/3}(x+3)^{2/3} = 0 \quad \Rightarrow \quad x = -3, 0.$$

Thus f has x-intercepts $(0,0)$ and $(-3,0)$.

10c. Since the domain of f is $(-\infty, \infty)$ there can be no vertical asymptotes. And since $f(x) \to \infty$ as $x \to \infty$, and $f(x) \to -\infty$ as $x \to -\infty$, there are no horizontal asymptotes either.

10d. Differentiating f gives

$$\begin{aligned} f'(x) &= x^{1/3} \cdot \frac{2}{3}(x+3)^{-1/3} + \frac{1}{3}x^{-2/3} \cdot (x+3)^{2/3} = \frac{2x^{1/3}}{3(x+3)^{1/3}} + \frac{(x+3)^{2/3}}{3x^{2/3}} \\ &= \frac{2x}{3x^{2/3}(x+3)^{1/3}} + \frac{x+3}{3x^{2/3}(x+3)^{1/3}} = \frac{3x+3}{3x^{2/3}(x+3)^{1/3}} = \frac{x+1}{x^{2/3}(x+3)^{1/3}} \end{aligned}$$

for any $x \neq -3, 0$. We see that $f' > 0$ if $x + 1 > 0$ and $x + 3 > 0$, which implies that $x > -1$; also $f' > 0$ if $x + 1 < 0$ and $x + 3 < 0$, which implies that $x < -3$. Thus f is increasing on $(-\infty, -3)$ and $(-1, \infty)$ by the Monotonicity Test. Since $f' < 0$ on $(-3, -1)$ we conclude that f is decreasing on this interval.

Now we find the critical points for f. Setting $f'(x) = 0$ gives $x = -1$, which is one critical point. As for x values for which $f'(x)$ does not exist, we have $x = -3, 0$, which are two more critical points. Since $f' > 0$ to the left of -3 and $f' < 0$ to the right of -3, by the First Derivative Test it follows that f has a local maximum at -3, with local maximum value of $f(-3) = 0$. Since $f' < 0$ to the left of -1 and $f' > 0$ to the right of -1, f has a local minimum at -1, with local minimum value of $f(-1) = -\sqrt[3]{4}$. Finally, since $f' > 0$ to the left and right of 0, there is no local extremum for f at 0.

10e. Next, we have

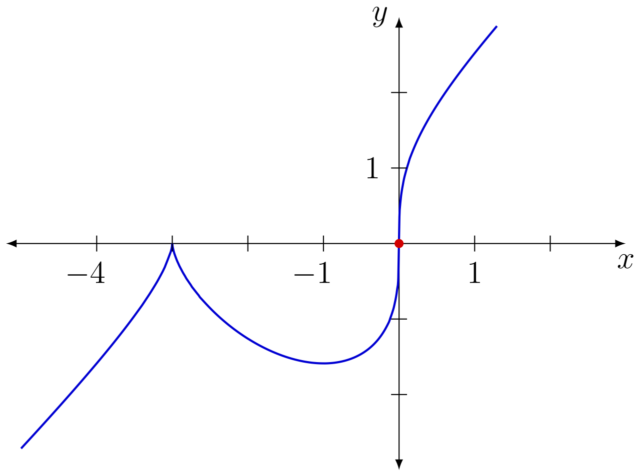
$$\begin{aligned} f''(x) &= \frac{(x^3+3x^2)^{1/3} - (x+1) \cdot \frac{1}{3}(x^3+3x^2)^{-2/3}(3x^2+6x)}{(x^3+3x^2)^{2/3}} \\ &= \frac{(x^3+3x^2) - (x+1)(x^2+2x)}{(x^3+3x^2)^{4/3}} = -\frac{2x}{(x^3+3x^2)^{4/3}} \end{aligned}$$

for all $x \neq -3, 0$. Since

$$(x^3+3x^2)^{4/3} = \left(\sqrt[3]{x^2(x+3)}\right)^4 > 0$$

for all $x \neq -3, 0$, we see that $f'' > 0$ on $(-\infty, -3) \cup (-3, 0)$ and $f'' < 0$ on $(0, \infty)$, and so by the Concavity Test f is concave up on $(-\infty, -3) \cup (-3, 0)$ and concave down on $(0, \infty)$. The point $(0, 0)$ is therefore an inflection point. At the point $(-3, 0)$ concavity does not change so there is no inflection point there.

10f. The inflection point is marked in red.



11. If x and y are the length and width of the garden, then $xy = 30$ and so $y = 30/x$. Meanwhile the combined area A of the garden and border is $(x+4)(y+2)$, or

$$A(x) = (x+4)\left(\frac{30}{x} + 2\right) = 38 + 2x + \frac{120}{x}.$$

The goal is to find x so that $A(x)$ is minimized. We have

$$A'(x) = 2 - \frac{120}{x^2},$$

and so if we set $A'(x) = 0$ we obtain

$$2 - \frac{120}{x^2} = 0 \;\Rightarrow\; 2x^2 - 120 = 0 \;\Rightarrow\; x^2 = 60 \;\Rightarrow\; x = \sqrt{60} = 2\sqrt{15}$$

(obviously we must have $x > 0$). Thus the length of the garden should be $x = 2\sqrt{15}$ m, and the width should be $y = 30/x = 30/(2\sqrt{15}) = \sqrt{15}$ m, in order to minimize A. That is, the garden should have dimensions $2\sqrt{15}$ m $\times$ $\sqrt{15}$ m.

12. Let L_1 and L_2 be the weaker and stronger light sources, respectively, and let I_1 and I_2 be their intensities. If p is the point on the line segment joining L_1 and L_2 that is a distance of x from L_1, then I_1 and I_2 may be characterized as functions of x:

$$I_1(x) = \frac{ks_1}{x^2} \quad \text{and} \quad I_2(x) = \frac{ks_2}{(12-x)^2},$$

where $k > 0$ is a constant of proportionality (dependent on what kind of unit is being used to quantify "intensity"), and $s_1, s_2 > 0$ are the "strengths" of L_1 and L_2. The total light intensity I at p is thus

$$I(x) = I_1(x) + I_2(x) = \frac{ks_1}{x^2} + \frac{2ks_1}{(12-x)^2}, \quad x \in (0, 12),$$

where we use the fact that $s_2 = 2s_1$.

We must find the global minimum for I, which is the minimum value $I(x)$ attains for $0 < x < 12$. We have

$$I'(x) = -\frac{2ks_1}{x^3} + \frac{4ks_1}{(12-x)^3} = \frac{2ks_1[2x^3 - (12-x)^3]}{x^3(12-x)^3},$$

and so from $I'(x) = 0$ we obtain the equation $2x^3 - (12-x)^3 = 0$, where

$$2x^3 - (12-x)^3 = 0 \;\Rightarrow\; 2x^3 = (12-x)^3 \;\Rightarrow\; x\sqrt[3]{x} = 12 - x \;\Rightarrow\; x = \frac{12}{1+\sqrt[3]{2}} := x^*,$$

which is approximately 5.31 and so is a critical point for I that lies in $(0, 12)$. There is no $x \in (0, 12)$ for which $I'(x)$ does not exist, so there are no other critical points in $(0, 12)$. Since I is continuous on $(0, 12)$ and

$$I'(4) = -\frac{3}{128}ks_1 < 0 \quad \text{and} \quad I'(8) = \frac{15}{256}ks_1 > 0,$$

we conclude by the Intermediate Value Theorem that $I' < 0$ on $(0, x^*)$, and $I' > 0$ on $(x^*, 12)$. By the First Derivative Test I has a local minimum at x^*, and since I is decreasing on $(0, x^*)$ and increasing on $(x^*, 12)$, we conclude that the local minimum at x^* is in fact a global minimum.

Therefore the intensity of light between L_1 and L_2 is weakest a distance of

$$\frac{12}{1+\sqrt[3]{2}} \approx 5.31 \text{ m}$$

from the weaker light source L_1.