MATH 140 EXAM #3 KEY (SUMMER II - 2010)

1. The linearization of f at 0 is simply the tangent line to the curve given by $f(x) = 1/\sqrt{2+x}$ at the point $(0, f(0)) = (0, 1/\sqrt{2})$. The slope of the line is figured from $f'(x) = -\frac{1}{2}(2+x)^{-3/2}$ as $f'(0) = -\frac{1}{2} \cdot 2^{-3/2} = -2^{-5/2}$. The point-slope formula gives an equation for the tangent line, $y - 2^{-1/2} = -2^{-5/2}(x-0)$, which simplifies as $y = -\frac{1}{4\sqrt{2}}x + \frac{1}{\sqrt{2}}$. Thus $L(x) = -\frac{1}{4\sqrt{2}}x + \frac{1}{\sqrt{2}}$, or approximately $L(x) = -0.1768x + 0.7071$.

2. We have the function $f(x) = \sin x$. The tangent line to the curve given by $f(x) = \sin x$ at the point $(0, f(0)) = (0, 0)$ will provide a reasonable linearization of the sine function for the purpose of estimating $\sin 1°$. The slope of the tangent line is $f'(0) = \cos(0) = 1$, which gives us an equation for the tangent line: $y = x$. That is, $L(x) = x$ is our linearization, and close to 0 we can expect the value of $L(x)$ to be fairly close to $\sin x$. The trick, however, is that we must work in radians: $\sin 1° = \sin(\pi/180) \approx L(\pi/180) = \pi/180$. This is a decent approximation, since $\pi/180 = 0.0174532925...$ while $\sin 1° = 0.0174524064...$

3a. $s'(t) = 12t^3 + 12t^2 - 12t = 12t(t^2 + t - 1)$, so $s'(t) = 0$ implies that $t = 0, \frac{-1 \pm \sqrt{5}}{2}$ are the critical numbers.

3b. $f'(x) = \frac{4}{5}x^{-1/5}(x-4)^2 + x^{4/5} \cdot 2(x-4) = 2(x-4)\left[x^{4/5} + \frac{2}{5}x^{-1/5}(x-4)\right] = \frac{2}{5}(x-4)(7x-8)x^{-1/5}$, so the critical numbers are $x = 0,\ 4,\ \frac{8}{7}$.

4a. $f'(x) = 4x^3 - 4x = 4x(x^2-1) = 4x(x-1)(x+1)$, so the critical numbers are $0,\ 1,\ -1$. We evaluate: $f(-2) = 11$, $f(-1) = 2$, $f(0) = 3$, $f(1) = 2$, $f(3) = 66$. Absolute maximum is $f(3) = 66$, and absolute minimum is $f(-1) = f(1) = 2$.

4b. $f'(x) = \cos x - \sin x$. Now, $f'(x) = 0 \Rightarrow \cos x = \sin x \Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4}$, which is the only solution that fits in the interval $\left[0, \frac{\pi}{3}\right]$. Now, $f(0) = 1$, $f\left(\frac{\pi}{4}\right) = \frac{2}{\sqrt{2}} = \sqrt{2}$, $f\left(\frac{\pi}{3}\right) = \sin\frac{\pi}{3} + \cos\frac{\pi}{3} = \frac{\sqrt{3}}{2} + \frac{1}{2} = \frac{\sqrt{3}+1}{2}$. Absolute maximum: $f\left(\frac{\pi}{4}\right) = \sqrt{2}$, absolute minimum: $f(0) = 1$.

5. f is continuous on $(-\infty, -2) \cup (-2, \infty)$, which contains the closed interval $[1, 4]$. Also f is differentiable on $(-\infty, -2) \cup (-2, \infty)$, which contains the open interval $(1, 4)$. By the Mean Value Theorem there exists some $c \in (1, 4)$ such that $f'(c) = \frac{f(4) - f(1)}{4 - 1} = \frac{1}{3}\left(\frac{2}{3} - \frac{1}{3}\right) = \frac{1}{9}$. Now, $f'(x) = \frac{2}{(x+2)^2}$, so $f'(c) = \frac{1}{9}$ implies that $\frac{2}{(c+2)^2} = \frac{1}{9}$, or $c = -2 \pm 3\sqrt{2}$. Now, notice that $-2 + 3\sqrt{2}$ lies in $(1, 4)$.

6a. $h'(x) = 15x^2(x^2-1)$, which should make clear that $h'(x) > 0$ on $(-\infty, -1) \cup (1, \infty)$ and $h'(x) < 0$ on $(-1, 0) \cup (0, 1)$; thus, $h(x)$ is increasing on $(-\infty, -1) \cup (1, \infty)$ and decreasing on $(-1, 0) \cup (0, 1)$.

6b. Local maximum is $h(-1) = 5$, and local minimum is $h(1) = 1$.

6c. $h''(x) = 60x^3 - 30x = 30x(\sqrt{2}x - 1)(\sqrt{2}x + 1)$, so $h''(x) > 0$ on $\left(-\frac{1}{\sqrt{2}}, 0\right) \cup \left(\frac{1}{\sqrt{2}}, \infty\right)$ and $h''(x) < 0$ on $\left(-\infty, -\frac{1}{\sqrt{2}}\right) \cup \left(0, \frac{1}{\sqrt{2}}\right)$. Therefore $h(x)$ is concave up on $\left(-\frac{1}{\sqrt{2}}, 0\right) \cup \left(\frac{1}{\sqrt{2}}, \infty\right)$ and concave down on $\left(-\infty, -\frac{1}{\sqrt{2}}\right) \cup \left(0, \frac{1}{\sqrt{2}}\right)$. Inflection points are at $x = -\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}$.

7a. $= \lim_{y\to\infty} \dfrac{-3y^2+2}{5y^2+4y} = -\dfrac{3}{5}.$

7b. $= \lim_{x\to-\infty} \dfrac{\sqrt{x^6(9-x^{-5})}}{x^3+1} = \lim_{x\to-\infty} \dfrac{|x|^3\sqrt{9-1/x^5}}{x^3+1} = \lim_{x\to-\infty} \dfrac{-x^3\sqrt{9-1/x^5}}{x^3+1} = \lim_{x\to-\infty} \dfrac{-\sqrt{9-1/x^5}}{1+1/x^3} = \dfrac{-\sqrt{9}}{1} = -3.$

7c. $= \lim_{x\to\infty} \left(\dfrac{\sqrt{x^4+6x^2}-x^2}{1} \cdot \dfrac{\sqrt{x^4+6x^2}+x^2}{\sqrt{x^4+6x^2}+x^2} \right) = \lim_{x\to\infty} \dfrac{6x^2}{\sqrt{x^4+6x^2}+x^2} = \lim_{x\to\infty} \dfrac{6x^2}{x^2\sqrt{1+6/x^2}+x^2}$

$= \lim_{x\to\infty} \dfrac{6}{\sqrt{1+6/x^2}+1} = \dfrac{6}{\sqrt{1+0}+1} = 3.$

8. A point on the line has coordinates $(x, 2x-9)$. The distance between $(x, 2x-9)$ and $(5, -2)$ is given by $D(x) = \sqrt{(x-5)^2 + ((2x-9)-(-2))^2} = \sqrt{5x^2 - 38x + 74}$. Now, $D'(x) = \frac{1}{2}(5x^2 - 38x + 74)^{-1/2} \cdot (10x - 38)$, and it's seen that $D'(x) = 0$ only when $x = \frac{38}{10}$. That is, the point $(\frac{19}{5}, 2(\frac{19}{5}) - 9) = (\frac{19}{5}, -\frac{7}{5})$ is the closest to $(5, -2)$.

9. $V = \pi r^2 h$, where V is regarded as a *constant*. We write $h = \dfrac{V}{\pi r^2}$ and thereby eliminate the variable h. The surface area S of the can is given by $S = \pi r^2 + 2\pi r h$, which then gives the function $S(r) = \pi r^2 + \dfrac{2V}{r}$. We want to find the minimum value for this function, so we obtain its derivative: $\dfrac{dS}{dr} = 2\pi r - \dfrac{2V}{r^2}$. The derivative does not exist when $r = 0$, but we can't have a can with zero radius so forget this critical number. Setting $\dfrac{dS}{dr} = 0$ and solving yields: $2\pi r - \dfrac{2V}{r^2} = 0$, or $r = \sqrt[3]{\dfrac{V}{\pi}}$. This is what we want. Dimensions of the can: radius of $\sqrt[3]{\dfrac{V}{\pi}}$, height of $\sqrt[3]{\dfrac{V}{\pi}}$ $\left(\text{put } r = \sqrt[3]{\dfrac{V}{\pi}} \text{ into } h = \dfrac{V}{\pi r^2}\right)$.

10. $\displaystyle\int \left(6x^{1/2} - x^{1/6}\right) dx = 6 \cdot \frac{2}{3}x^{3/2} - \frac{6}{7}x^{7/6} + C = 4x^{3/2} - \frac{6}{7}x^{7/6} + C.$

11. $f'(x) = \displaystyle\int f''(x)dx = 5x^4 + 4x^3 + 4x + C \;\Rightarrow\; f(x) = \int f'(x)dx = x^5 + x^4 + 2x^2 + Cx + D$. Now, $f(0) = 8$ implies that $D = 8$, giving us $f(x) = x^5 + x^4 + 2x^2 + Cx + 8$. Next, $f(1) = 5$ gives $1 + 1 + 2 + C + 8 = 5$, or $C = -7$. Therefore $f(x) = x^5 + x^4 + 2x^2 - 7x + 8$.

12. Let $f(x) = 4x^5 + x^3 + 2x + 1$, which is a function that is continuous and differentiable on $(-\infty, \infty)$. Now, $f(-1) = -6 < 0$ and $f(0) = 1 > 0$, so by the Intermediate Value Theorem there exists some $c \in (-1, 0)$ such that $f(c) = 0$ (i.e. $4c^5 + c^3 + 2c + 1 = 0$). This demonstrates that the equation has *at least one* real root.

Now, suppose f has two real roots c_1 and c_2. Then $f(c_1) = 0$ and $f(c_2) = 0$. By the Mean Value Theorem (or just apply its corollary, Rolle's Theorem) there exists some number b between c_1 and c_2 such that $f'(b) = 0$. Thus $20b^4 + 3b^2 + 2 = 0$. But we can see that we must have $f'(x) \geq 2$ for all $x \in (-\infty, \infty)$ since $20x^4 \geq 0$ and $3x^2 \geq 0$ always hold for $f'(x) = 20x^4 + 3x^2 + 2$. So $f'(b) = 0$ for some real number b is a contradiction. Hence f cannot have two real roots.

Therefore f must have *exactly* one real root. ■