# Math 140 Exam #3 Key (Summer 2020)

**1** Set $f'(x) = 12x^2 - 42x + 36 = 0$ to get $x = \frac{3}{2}, 2$. These are the critical points. Evaluate: $f(1) = 19$, $f(3) = 27$, $f(3/2) = 20.25$, $f(2) = 20$. Absolute maximum is $f(3) = 27$, absolute minimum is $f(1) = 19$.

**2** $f$ is continuous and differentiable on $(-\infty, -2) \cup (-2, \infty)$, so is continuous on $[-1, 2]$ and differentiable on $(-1, 2)$. Mean Value Theorem applies: there exists some $c \in (-1, 2)$ such that

$$f'(c) = \frac{f(2) - f(-1)}{2 - (-1)} = \frac{1}{2}.$$

We have $f'(x) = 2/(x+2)^2$, so $f'(c) = 1/2$ implies $2/(c+2)^2 = 1/2$. Solve to get $c = 0, -4$. Since $0 \in (-1, 2)$ the theorem is confirmed.

**3** First we have

$$f'(x) = 2x\sqrt{9 - x^2} - \frac{x^3}{\sqrt{9 - x^2}},$$

so since

$$f'(x) > 0 \;\Rightarrow\; 2x\sqrt{9 - x^2} - \frac{x^3}{\sqrt{9 - x^2}} > 0 \;\Rightarrow\; 2x(9 - x^2) > x^3$$

(multiply by $\sqrt{9 - x^2}$), it follows that $f'(x) > 0$ if and only if

$$3x(\sqrt{6} - x)(\sqrt{6} + x) > 0.$$

This has solution set $(-3, -\sqrt{6}) \cup (0, \sqrt{6})$, while $f'(x) < 0$ has solution set $(-\sqrt{6}, 0) \cup (\sqrt{6}, 3)$. Therefore $f$ is increasing on $(-3, -\sqrt{6})$ and $(0, \sqrt{6})$, and decreasing on $(-\sqrt{6}, 0)$ and $(\sqrt{6}, 3)$.

**4a** Domain is $\mathbb{R}$, while the only intercept is $(0, 0)$.

**4b** There's only the horizontal asymptote $y = 0$.

**4c** Here

$$f'(x) = \frac{12(2 - x^2)}{(x^2 + 2)^2},$$

so $f'(x) = 0$ if and only if $x = \pm\sqrt{2}$. These are the critical points.

**4d** $f'(x) < 0$ iff $2 - x^2 = 0$ iff $x \in (-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$, while $f'(x) > 0$ iff $x \in (-\sqrt{2}, \sqrt{2})$. Hence $f$ is decreasing on $(-\infty, -\sqrt{2})$, $(\sqrt{2}, \infty)$, and increasing on $(-\sqrt{2}, \sqrt{2})$. By the First Derivative Test $f(-\sqrt{2}) = -3\sqrt{2}$ is a local minimum and $f(\sqrt{2}) = 3\sqrt{2}$ is a local maximum.

**4e** Differentiating $f'(x)$ gives

$$f''(x) = \frac{24x(x^2 - 6)}{(x^2 + 2)^3}$$

So $f''(x) > 0$ iff $24x(x^2-6) > 0$ iff $x \in (-\sqrt{6}, 0) \cup (\sqrt{6}, \infty)$. We conclude that $f$ is concave up on $(-\sqrt{6}, 0)$, $(\sqrt{6}, \infty)$, and concave down on $(-\infty, -\sqrt{6})$, $(0, \sqrt{6})$. Inflection points are $(-\sqrt{6}, f(-\sqrt{6}))$, $(0, f(0))$, $(\sqrt{6}, f(\sqrt{6}))$.

**5** Let $x$ be the length of one leg of the right triangle, so that $1000 - x$ is the length of the other side. Area of triangle is $A(x) = \frac{1}{2}x(1000-x)$. Setting $A'(x) = 500 - x = 0$ gives $x = 500$. Area of triangle is maximized when the lengths of the two legs is 500.

**6** Suppose $d$ is the distance traveled by the driver. This is a constant here, whereas the velocity $v$ is the variable. Cost $C$ as a function of $v$ is

$$C(v) = \frac{15d}{v} + \frac{2.50d}{10 - 0.07v}.$$

Then

$$C'(v) = -\frac{15d}{v^2} - \frac{2.50d}{(10-0.07v)^2}(-0.07),$$

and setting $C'(v) = 0$ gives

$$\frac{0.175}{(10-0.07v)^2} = \frac{15}{v^2},$$

which has solutions

$$v = \frac{-21 \pm \sqrt{1050}}{0.2030}.$$

The choice of "$-$" gives a negative velocity, which we must discount. The "+" option gives $v \approx 56$ miles per hour, which is the velocity that minimizes cost.

Note that $C'(v)$ is undefined at $v = 0$ and $v \approx 142.9$, but since neither of these values is in the domain of $C(v)$ they are not critical points. (And anyway it's not reasonable for the driver to go 0 mph or 143 mph.)

**7** Set $f(x) = \sqrt[4]{x}$, so $f'(x) = \frac{1}{4}x^{-3/4}$, and hence $f'(81) = \frac{1}{108}$. Thus the tangent line to $f(x) = \sqrt[4]{x}$ at $(81, f(81)) = (81, 3)$ has slope $\frac{1}{108}$, resulting in the equation $y = \frac{1}{108}x + \frac{9}{4}$. Now,

$$\sqrt[4]{85} = f(85) \approx \frac{1}{108}(85) + \frac{9}{4} = 3.\overline{037}.$$

**8a** The limit could be figured by factoring out $\sqrt{x}$ from the numerator and denominator and reducing. But with L'Hôpital's Rule we have

$$\lim_{x\to 0^+} \frac{x - 3\sqrt{x}}{x - \sqrt{x}} \overset{\text{LR}}{=} \lim_{x\to 0^+} \frac{1 - \dfrac{3}{2\sqrt{x}}}{1 - \dfrac{1}{2\sqrt{x}}} = \lim_{x\to 0^+} \frac{2\sqrt{x} - 3}{2\sqrt{x} - 1} = \frac{2\sqrt{0} - 3}{2\sqrt{0} - 1} = 3.$$

**8b** From the identity $\sin(2u) = 2\sin u \cos u$ we get $\csc x = \frac{1}{2}\csc(x/2)\sec(x/2)$, which will help make things a bit easier:

$$\begin{aligned}\lim_{x\to\pi^-}\frac{\csc x + x}{\tan(x/2)} &= \lim_{x\to\pi^-}\frac{\frac{1}{2}\csc(x/2)\sec(x/2)+x}{\tan(x/2)}\cdot\frac{\cos(x/2)}{\cos(x/2)}\\ &= \lim_{x\to\pi^-}\frac{\frac{1}{2}\csc(x/2)+x\cos(x/2)}{\sin(x/2)}\\ &= \frac{\frac{1}{2}\csc(\pi/2)+x\cos(\pi/2)}{\sin(\pi/2)} = \frac{1}{2}.\end{aligned}$$

L'Hôpital's Rule makes things worse for this problem.

**8c** This one is like one I did in the lecture: L'Hôpital's Rule leads to nothing but messes, so we don't use it here. Multiply numerator and denominator by $\sqrt{x^2+x}+x$ to get

$$\lim_{x\to\infty}\frac{x}{\sqrt{x^2+x}+x} = \lim_{x\to\infty}\frac{1}{\sqrt{1+1/x}+1} = \frac{1}{2}.$$

**9a** $\frac{1}{3}y^3 + 4y^{-1/2} - 2y^{-1} + C.$

**9b** $8\sin\theta - \cos\theta + C.$

**10** We have $h(t) = 4\sin t + 4\cos t + C$, so

$$0 = h(\pi/2) = 4\sin\frac{\pi}{2} + 4\cos\frac{\pi}{2} + C = 4 + C,$$

implying $C = -4$, and therefore

$$h(t) = 4\sin t + 4\cos t - 4.$$