Math 140 Exam #2 Key (Summer 2020)

1a $r'(t) = t^{-2/3} - 6t^7 - 1$

1b Quotient Rule:

$$G'(\ell) = \frac{(\sqrt{\ell}+2)(2) - (2\ell - 1)(\frac{1}{2}\ell^{-1/2})}{(\sqrt{\ell}+2)^2} = \frac{2\ell + 8\sqrt{\ell} + 1}{2\sqrt{\ell}(\sqrt{\ell}+2)^2}.$$

1c Product Rule: $\sin x \cos x + x\cos^2 x - x\sin^2 x$

1d Quotient Rule:

$$y' = \frac{(1+\csc x)(-\csc^2 x) - (\cot x)(-\csc x \cot x)}{(1+\csc x)^2} = \frac{\csc x \cot^2 x - \csc^2 x - \csc^3 x}{(1+\csc x)^2}.$$

2 We need $f'(2) = 4$ and we have $f'(x) = 2x + A$, so let $A = 0$. Now, $f(x) = x^2 + B$ must have tangent line $y = 4x + 2$ at the point $(2, f(2)) = (2, B+4)$, and we find its tangent line to in fact be $y = 4x + B - 4$. Thus $B - 4 = 2$ is required, or $B = 6$. The desired function f is therefore given by $f(x) = x^2 + 6$.

3a $y' = 8(2x^6 + x)^7(12x^5 + 1)$.

3b $y' = \sec^2(\sqrt{x}\,) \cdot (\sqrt{x}\,)' = \dfrac{\sec^2\sqrt{x}}{2\sqrt{x}}$.

3c $h'(x) = -20\sec^4(\cos 5x)\tan(\cos 5x)\sin 5x$.

4 Differentiating both sides of the equation with respect to x gives

$$3(xy+1)^2(xy' + y) = 1 - 2yy'.$$

Now just solve for y':

$$y' = \frac{1 - 3y(xy+1)^2}{3x(xy+1)^2 + 2y}.$$

5 Differentiating both sides of the equation with respect to x:

$$-(1-y')\sin(x-y) + y'\cos y = 0.$$

Solve for y':

$$y' = \frac{\sin(x-y)}{\sin(x-y) + \cos y}.$$

At the point $(x, y) = (\pi/2, \pi/4)$ we then find that

$$y' = \frac{\sin \pi/4}{\sin \pi/4 + \cos \pi/4} = \frac{1}{2}.$$

This is the slope of the tangent line at $(\pi/2, \pi/4)$, and so the equation is $y = \frac{1}{2}x$.

6 With b and h the base and height of the triangle, respectively, the area is $A = \frac{1}{2}bh$. Differentiating with respect to time t yields

$$\frac{dA}{dt} = \frac{1}{2}\left(b\frac{dh}{dt} + h\frac{db}{dt}\right). \tag{1}$$

When area is $A = 150$ cm^2 and height is $h = 12$ cm we find the base to be $b = 2A/h = 25$ cm. Putting all known quantities into (1) gives

$$3 = \frac{1}{2}\left(25(-2) + 12\frac{db}{dt}\right),$$

or $db/dt = 4\frac{2}{3}$ cm/min.

7 Let h be the water depth, so the water occupies a space that is an inverted cone with radius $r = h/2$ (using similar triangles). With the formula $V = \frac{\pi}{3}r^2h$ we find the water volume to be $V = \frac{\pi}{12}h^3$. Now implicit differentiation gives

$$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt},$$

and hence

$$\frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt}.$$

We're given that $dV/dt = -2$ ft^3/s, so when $h = 3$ ft we have

$$\left.\frac{dh}{dt}\right|_{h=3} = \frac{4}{\pi \cdot 3^2}(-2) = -\frac{8}{9\pi} \text{ ft/s}.$$