Math 140 Exam #1 Key (Summer 2020)

1 We have

$$\lim_{x\to -1} F(x) = 3, \quad \lim_{x\to -2} F(x) = 1, \quad \lim_{x\to 1^-} F(x) = 2, \quad \lim_{x\to 1^+} F(x) = 4, \quad \lim_{x\to 3^+} F(x) = \text{DNE}.$$

2a Simply reduce the fraction first:

$$\lim_{x\to -b} \frac{(x+b)^6 + (x+b)^9}{4} = 0.$$

2b Multiply top and bottom of fraction by $5(5+h)$:

$$\lim_{h\to 0} \frac{5-(5+h)}{5h(5+h} = \lim_{h\to 0} \frac{-1}{5(5+h)} = -\frac{1}{25}.$$

2c Factor the bottom (or multiply top and bottom by $\sqrt{x}+8$):

$$\lim_{x\to 64} \frac{\sqrt{x}-8}{(\sqrt{x}-8)(\sqrt{x}+8)} = \lim_{x\to 64} \frac{1}{\sqrt{x}+8} = \frac{1}{\sqrt{64}+8} = \frac{1}{16}.$$

2d Factor the bottom:

$$\lim_{x\to 0} \frac{1-\cos x}{(\cos x - 2)(\cos x - 1)} = \lim_{x\to 0} \frac{-1}{\cos x - 2} = \frac{-1}{\cos 0 - 2} = 1.$$

3 Since

$$\lim_{x\to -3^-} G(x) = \lim_{x\to -3^-} (3x-4k) = -9-4k \quad \text{and} \quad \lim_{x\to -3^+} G(x) = \lim_{x\to -3^+} (x+9) = 6,$$

the limit $\lim_{x\to -3} G(x)$ can only exist if $-9-4k = 6$, which only happens if $k = -\frac{15}{4}$. Then $\lim_{x\to -3} G(x) = 6$.

4 All equal $-\infty$.

5 For $x \neq \frac{1}{2}$,

$$f(x) = \frac{1}{x(x+3)},$$

and so f has vertical asymptotes $x=0$ and $x=-3$. We find that

$$\lim_{x\to 0^-} f(x) = -\infty, \quad \lim_{x\to 0^+} f(x) = +\infty, \quad \lim_{x\to -3^-} f(x) = +\infty, \quad \lim_{x\to -3^+} f(x) = -\infty,$$

so $\lim_{x\to 0} f(x)$ and $\lim_{x\to -3} f(x)$ do not exist.

6 Divide top and bottom by x^2, then evaluate to get $-\frac{4}{7}$.

7 In general,

$$f(x) = \frac{4x^3+1}{2x^3+\sqrt{16x^6+1}} = \frac{4x^3+1}{2x^3+|x|^3\sqrt{16+x^{-6}}},$$

so

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{4x^3+1}{2x^3+x^3\sqrt{16+x^{-6}}} = \lim_{x\to\infty} \frac{4+x^{-3}}{2+\sqrt{16+x^{-6}}} = \frac{4}{2+\sqrt{16}} = \frac{2}{3},$$

and

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{4x^3+1}{2x^3-x^3\sqrt{16+x^{-6}}} = \lim_{x\to-\infty} \frac{4+x^{-3}}{2-\sqrt{16+x^{-6}}} = \frac{4}{2-\sqrt{16}} = -2.$$

Horizontal asymptotes are $y = \frac{2}{3}$ and $y = -2$.

8 Continuity from the left at -1 requires that $\lim_{x\to-1^-} h(x) = h(-1)$. Since

$$\lim_{x\to-1^-} h(x) = \lim_{x\to-1^-} (2x^2 - x) = 3,$$

and $h(-1) = s$, we set $s = 3$ to get continuity from the left at -1.

Continuity from the right at -1 requires $\lim_{x\to-1^+} h(x) = h(-1)$. Since

$$\lim_{x\to-1^+} h(x) = \lim_{x\to-1^+} (3x-5) = -8,$$

and $h(-1) = s$, we set $s = -8$ to get continuity from the right at -1.

9 Let $\epsilon > 0$. Choose $\delta = \epsilon/4$. Suppose x is such that $0 < |x+2| < \delta$. Then $|x+2| < \epsilon/4$, and since

$$|x+2| < \frac{\epsilon}{4} \quad\Rightarrow\quad |4x+8| < \epsilon \quad\Rightarrow\quad |(4x+9)-1| < \epsilon,$$

we conclude that $4x+9 \to 1$ as $x \to -2$.

10a By definition,

$$f'(1) = \lim_{t\to1} \frac{f(t)-f(1)}{t-1} = \lim_{t\to1} \frac{\sqrt{2t-1}-1}{t-1} = \lim_{t\to1} \frac{2t-2}{(t-1)\left(\sqrt{2t-1}+1\right)} = \lim_{t\to1} \frac{2}{\sqrt{2t-1}+1} = 1.$$

10b Slope of the tangent line at $(1,1)$ is $f'(1) = 1$, so line is $y = x$.

11 We have

$$\begin{aligned} f'(x) &= \lim_{t\to x} \frac{f(t)-f(x)}{t-x} = \lim_{t\to x} \frac{\dfrac{1}{3t-4} - \dfrac{1}{3x-4}}{t-x} = \lim_{t\to x} \frac{-3(t-x)}{(t-x)(3t-4)(3x-4)} \\ &= \lim_{t\to x} \frac{-3}{(3t-4)(3x-4)} = -\frac{3}{(3x-4)^2}. \end{aligned}$$