Math 140 Exam #3 Key (Summer 2019)

1 We have $f'(x) = 4x^3 - 12x^2 + 8x = 4x(x-1)(x-2)$, so $f'(x) = 0$ when $x = 0, 1, 2$. These are the critical points. We evaluate:

$$f(-1) = 9, \quad f(0) = 0, \quad f(1) = 1, \quad f(2) = 0, \quad f(3) = 9.$$

The absolute minimum value of f on $[-1, 3]$ is $f(0) = f(2) = 0$, and the absolute maximum value is $f(-1) = f(3) = 9$.

2a Domain is $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$. The only intercept is $(0, 0)$.

2b Horizontal asymptote: $y = 1$. Vertical asymptotes: $x = \pm 2$.

2c Since

$$f'(x) = -\frac{8x}{(x^2-4)^2},$$

the only critical point of f is $x = 0$.

2d For x in the domain of f, we have $f'(x) > 0$ for $x < 0$, and $f'(x) < 0$ for $x > 0$. By the Monotonicity Test f is increasing on $(-\infty, -2) \cup (-2, 0)$, and decreasing on $(0, 2) \cup (2, \infty)$. By the First Derivative Test f has a local maximum at $(0, 0)$.

2e Here

$$f''(x) = \frac{24x^2 + 32}{(x^2-4)^3},$$

so $f''(x) < 0$ for $-2 < x < 2$, and $f''(x) > 0$ for $x < -2$ and $x > 2$. Therefore, by the Concavity Test, f is concave down on $(-2, 2)$, and concave up on $(-\infty, -2)$ and $(2, \infty)$. There are no inflection points.

3 A point on $y = 3x$ has the form $(x, 3x)$, and this point's distance from $(50, 0)$ is

$$d(x) = \sqrt{(x-50)^2 + (3x)^2} = \sqrt{10x^2 - 100x + 2500}.$$

We can minimize $d^2(x)$ a bit easier than $d(x)$ itself. Define

$$D(x) = d^2(x) = 10x^2 - 100x + 2500.$$

Then $D'(x) = 0$ implies $20x - 100 = 0$, giving $x = 5$. The point on $y = 3x$ closest to $(50, 0)$ is therefore $(5, 15)$. Distance between these points is $\sqrt{45^2 + 15^2}$.

4 Let x be as in the figure below (i.e. x is the distance between the nearest point on the shore to the island and the point where the cable will meet the shore). Cost function is:

$$C(x) = \left(\frac{\$2400}{\text{km}}\right)\left(\sqrt{x^2 + 3.5^2}\ \text{km}\right) + \left(\frac{\$1200}{\text{km}}\right)(8 - x\ \text{km}) = 2400\sqrt{x^2 + \frac{49}{4}} + 1200(8 - x).$$

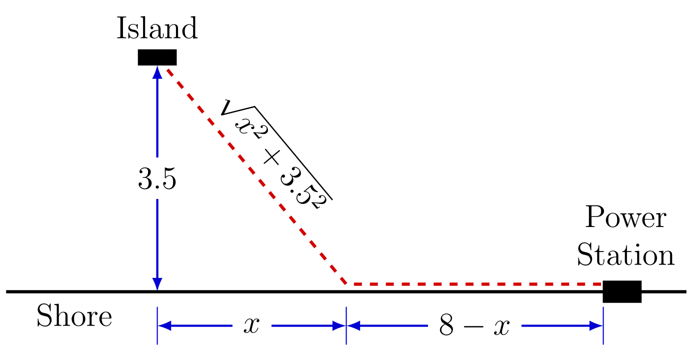
We take the derivative:

$$C'(x) = \frac{2400x}{\sqrt{x^2 + 49/4}} - 1200.$$

Note that there is no x value for which $C'(x)$ does not exist. On the other hand,

$$C'(x) = 0 \;\Rightarrow\; \frac{2x}{\sqrt{x^2+49/4}} - 1 = 0 \;\Rightarrow\; 2x = \sqrt{x^2 + \frac{49}{4}} \;\Rightarrow\; 4x^2 = x^2 + \frac{49}{4}$$
$$\Rightarrow\; x^2 = \frac{49}{12} \;\Rightarrow\; x = \sqrt{\frac{49}{12}} = \frac{7}{2\sqrt{3}} = \frac{7\sqrt{3}}{6}.$$

Thus, if the cable meets the shore at a point about $8 - 7\sqrt{3}/6$ km to the left of the power station, cost will be minimized.



5 We have $g'(t) = 1/\sqrt{2t+9}$. The linear approximation in question is the line through $(-4, g(-4)) = (-4, 1)$ with slope $g'(-4) = 1$, which has equation $y = t + 5$, and so the function $L(t) = t + 5$ is the linear approximation (or "linearization") of $g(t)$ at $t = -4$.

6 By the Mean Value Theorem there exists some $c \in (-2, 14)$ such that

$$f'(c) = \frac{f(14) - f(-2)}{14 - (-2)} = \frac{7 - f(-2)}{16}.$$

Since $f'(c) \le 10$ is required, we must have $7 - f(-2) \le 160$, or $f(-2) \ge -153$.

7a We have

$$\lim_{x\to\pi/2^-}\left(\frac{\pi}{2} - x\right)\sec x = \lim_{x\to\pi/2^-} \frac{\frac{\pi}{2} - x}{\cos x} = \lim_{x\to\pi/2^-} \frac{-1}{-\sin x} = \frac{1}{\sin(\pi/2)} = 1.$$

7b Using L'Hôpital's Rule three times:

$$\lim_{x\to 0}\frac{\tan x - x}{x^3} = \lim_{x\to 0}\frac{\sec^2 x - 1}{3x^2} = \lim_{x\to 0}\frac{2\sec^2 x \tan x}{6x} = \lim_{x\to 0}\frac{2\sec^4 x + 4\tan^2 x \sec^2 x}{6}$$
$$= \frac{2\sec^4(0) + 4\tan^2(0)\sec^2(0)}{6} = \frac{2\cdot 1^4 + 4\cdot 0^2 \cdot 1^2}{6} = \frac{1}{3}.$$

8a $\int \left(\frac{5}{t^2} + 4t^2 \right) dt = \int (5t^{-2} + 4t^2) dt = -\frac{5}{t} + \frac{4}{3} t^3 + C.$

8b $\int (\sin 2y - \cos 6y) dy = -\frac{1}{2} \cos 2y - \frac{1}{6} \sin 6y + C.$