Math 140 Exam #2 Key (Summer 2019)

1a Since $\sqrt{t} = t^{1/2}$, $s'(t) = 2t^{-1/2} - t^3 + 1$.

1b Quotient Rule:

$$f'(x) = \frac{(x^2-1)(4x^3) - (x^4+1)(2x)}{(x^2-1)^2} = \frac{4x^5 - 4x^3 - 2x^5 - 2x}{(x^2-1)^2} = \frac{2x^5 - 4x^3 - 2x}{(x^2-1)^2}.$$

1c Product Rule: $y' = \sin x \sec^2 x + \cos x \tan x = \tan x \sec x + \sin x$.

1d Quotient Rule:

$$y' = \frac{(1+\sin x)(-2\sin x) - (2\cos x)(\cos x)}{(1+\sin x)^2} = -\frac{2 + 2\sin x}{(1+\sin x)^2} = -\frac{2}{1+\sin x}.$$

2 First we get $f'(x) = 6x^2 - 6x - 12$. Setting $f'(x) = 60$ gives $6x^2 - 6x - 12 = 60$, or $x^2 - x - 12 = 0$, which factors as $(x-4)(x+3) = 0$. Solutions are $x = -3, 4$. Points on the graph of f with slope 60 are therefore $(-3, -41)$ and $(4, 36)$.

3a $y' = 16(4 - 15x^4)(4x - 3x^5)^{15}$.

3b $y' = \sec^2(\sqrt{x}\,) \cdot (\sqrt{x}\,)' = \dfrac{1}{2\sqrt{x}} \sec^2 \sqrt{x}$.

3c $h'(x) = 4\sin^3(\cos 7x) \cdot \cos(\cos 7x) \cdot (-\sin 7x) \cdot 7 = -28\cos(\cos 7x)\sin^3(\cos 7x)\sin 7x$.

4 By the Quotient Rule:

$$3x^2 = \frac{(x-y)(1+y') - (x+y)(1-y')}{(x-y)^2} \quad\Rightarrow\quad y' = \frac{3x^2(x-y)^2 + 2y}{2x}.$$

Alternative: rewrite equation as $x^4 - x^3y = x + y$ to avoid having to use the Quotient Rule, if desired. Now differentiate with respect to x:

$$4x^3 - 3x^2y - x^3y' = 1 + y' \quad\Rightarrow\quad 4x^3 - 3x^2y - 1 = x^3y' + y' = (x^3+1)y' \quad\Rightarrow\quad y' = \frac{4x^3 - 3x^2y - 1}{x^3+1}.$$

5 Implicit differentiation gives

$$y^{5/2} + \tfrac{5}{2}xy^{3/2}y' + \tfrac{3}{2}x^{1/2}y + x^{3/2}y' = 0,$$

and so

$$y' = -\frac{y^{5/2} + \frac{3}{2}x^{1/2}y}{x^{3/2} + \frac{5}{2}xy^{3/2}} = -\frac{2y^{5/2} + 3x^{1/2}y}{2x^{3/2} + 5xy^{3/2}}$$

At the point $(x, y) = (4, 1)$ we have

$$y' = -\frac{2(1)^{5/2} + 3(4)^{1/2}(1)}{2(4)^{3/2} + 5(4)(1)^{3/2}} = -\frac{2 + 3(2)}{2(8) + 5(4)} = -\frac{8}{36} = -\frac{2}{9},$$

which is the slope of the tangent line. Equation of tangent line is therefore

$$y - 1 = -\tfrac{2}{9}(x - 4) \quad \Rightarrow \quad y = -\tfrac{2}{9}x + \tfrac{17}{9} \quad \text{or} \quad 2x + 9y = 17$$

6 With b and h the base and height of the triangle, respectively, the area is $A = \frac{1}{2}bh$. Differentiating with respect to time t yields

$$\frac{dA}{dt} = \frac{1}{2}\left(b\frac{dh}{dt} + h\frac{db}{dt}\right). \tag{1}$$

When area is $A = 150$ cm^2 and height is $h = 12$ cm we find the base to be $b = 2A/h = 25$ cm. Putting all known quantities into (1) gives

$$2 = \frac{1}{2}\left(25(-1) + 12\frac{db}{dt}\right),$$

or $db/dt = 2\frac{5}{12}$ cm/min.

7 The ground, wall, and ladder form a right triangle with hypotenuse of length 13. If x is the distance between the wall and the foot of the ladder, and y is the distance between the ground and the top of the ladder, then $x^2 + y^2 = 13^2$. Both x and y are functions of time t, and so differentiating with respect to t yields

$$2x(t)x'(t) + 2y(t)y'(t) = 0 \quad \Rightarrow \quad y'(t) = -\frac{x(t)x'(t)}{y(t)}.$$

We're given that $x'(t) = 0.5$ ft/s for all $t \geq 0$, and of course we also have $y(t) = \sqrt{169 - x^2(t)}$. Thus

$$y'(t) = -\frac{x(t)}{2\sqrt{169 - x^2(t)}}.$$

Now, at the time t when $x(t) = 5$ ft, we obtain

$$y'(t) = -\frac{5}{2\sqrt{169 - 5^2}} = -\frac{5}{24};$$

that is, at the time t when the foot of the ladder is 5 ft from the wall, the top of the ladder is sliding down the wall at a rate of $-5/24$ ft/s.