Math 140 Exam #1 Key (Summer 2019)

1 We have

$$\lim_{x\to 3^-}\lfloor x\rfloor = 2,\quad \lim_{x\to 3^+}\lfloor x\rfloor = 3,\quad \lim_{x\to -6^-}\lfloor x\rfloor = -7,\quad \lim_{x\to -6^+}\lfloor x\rfloor = -6,\quad \lim_{x\to 0.9}\lfloor x\rfloor = 0.$$

2a Simply reduce the fraction first:

$$\lim_{x\to b}\left[-(x-b)^{39}+1\right] = 1.$$

2b $\displaystyle\lim_{w\to 1}\frac{1-w}{w^2-w} = \lim_{w\to 1}\left(-\frac{1}{w}\right) = -1.$

2c $\displaystyle\lim_{x\to 4}\frac{3(x-4)\sqrt{x+5}\cdot(3+\sqrt{x+5})}{9-(x+5)} = -\lim_{x\to 4}3\sqrt{x+5}\left(3+\sqrt{x+5}\,\right) = -54.$

2d $\displaystyle\lim_{x\to 0}\frac{\cos x-1}{1-\cos^2 x} = \lim_{x\to 0}\frac{-(1-\cos x)}{(1-\cos x)(1+\cos x)} = \lim_{x\to 0}\frac{-1}{1+\cos x} = -\frac{1}{2}.$

3 Since

$$\lim_{x\to -2^-}p(x) = \lim_{x\to -2^-}(3x+r) = -6+r \quad\text{and}\quad \lim_{x\to -2^+}p(x) = \lim_{x\to -2^+}(x-12) = -14,$$

the limit $\lim_{x\to -2}p(x)$ can only exist if $-6+r=-14$, which only happens if $r=-8$. Then $\lim_{x\to -2}p(x) = -14$.

4 We have

$$\lim_{x\to 3^-}\frac{-5}{(x-3)^3} = +\infty,\quad \lim_{t\to -2^+}\frac{t(t-2)(t-3)}{t^2(t-2)(t+2)} = \lim_{t\to -2^+}\frac{t-3}{t(t+2)} = +\infty,\quad \lim_{\theta\to 0^-}\csc\theta = -\infty.$$

5 Since

$$f(x) = \frac{x+1}{x(x-2)^2},$$

f has vertical asymptotes $x=0$ and $x=2$. We find that

$$\lim_{x\to 0^-}f(x) = -\infty,\quad \lim_{x\to 0^+}f(x) = +\infty,\quad \lim_{x\to 2^-}f(x) = +\infty,\quad \lim_{x\to 2^+}f(x) = +\infty,$$

so $\lim_{x\to 2}f(x) = +\infty$ and $\lim_{x\to 0}f(x)$ does not exist.

6 Divide by the highest power of x in the denominator:

$$\lim_{x\to\infty}\frac{4x^2-7}{8x^2+5x+2}\cdot\frac{x^{-2}}{x^{-2}} = \lim_{x\to\infty}\frac{4-7x^{-2}}{8+5x^{-1}+2x^{-2}} = \frac{4-0}{8+0+0} = \frac{1}{2}.$$

7 Recall that $\sqrt{x^2} = |x|$, so $\sqrt{x^2} = x$ if $x \geq 0$ and $\sqrt{x^2} = -x$ if $x < 0$. Now, since $x \to \infty$ implies $x > 0$,

$$\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{2x+1} = \lim_{x\to\infty} \frac{\sqrt{x^2(1+x^{-2})}}{2x+1} = \lim_{x\to\infty} \frac{x\sqrt{1+x^{-2}}}{2x+1} = \lim_{x\to\infty} \frac{\sqrt{1+x^{-2}}}{2+x^{-1}} = \frac{\sqrt{1+0}}{2+0} = \frac{1}{2};$$

and since $x \to -\infty$ implies $x < 0$,

$$\lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{2x+1} = \lim_{x\to-\infty} \frac{\sqrt{x^2(1+x^{-2})}}{2x+1} = \lim_{x\to\infty} \frac{-x\sqrt{1+x^{-2}}}{2x+1} = \lim_{x\to\infty} \frac{-\sqrt{1+x^{-2}}}{2+x^{-1}} = -\frac{1}{2}.$$

Horizontal asymptotes are $y = \frac{1}{2}$ and $y = -\frac{1}{2}$.

8 Continuity from the left at 1 requires that $\lim_{x\to1^-} g(x) = g(1)$. Since

$$\lim_{x\to1^-} g(x) = \lim_{x\to1^-} (x^2 - 2x) = -1,$$

and $g(1) = a$, we set $a = -1$ to get continuity from the left at 1.

Continuity from the right at 1 requires $\lim_{x\to1^+} g(x) = g(1)$. Since

$$\lim_{x\to1^+} g(x) = \lim_{x\to1^+} (3x + 9) = 12,$$

and $g(1) = a$, we set $a = 12$ to get continuity from the right at 1.

9 Let $\epsilon > 0$. Choose $\delta = \epsilon/4$. Suppose $x \in \mathbb{R}$ is such that $0 < |x - 8| < \delta$. Then $|x - 8| < \epsilon/4$, and since

$$|x-8| < \frac{\epsilon}{4} \quad\Rightarrow\quad 4|x-8| < \epsilon \quad\Rightarrow\quad |4x-32| < \epsilon \quad\Rightarrow\quad |(4x-5)-27| < \epsilon,$$

we conclude that $4x - 5 \to 27$ as $x \to 8$.

10a By definition,

$$f'(1) = \lim_{t\to1} \frac{f(t)-f(1)}{t-1} = \lim_{t\to1} \frac{\sqrt{t+3}-2}{t-1} = \lim_{t\to1} \frac{(t+3)-4}{(t-1)\big(\sqrt{t+3}+2\big)} = \lim_{t\to1} \frac{1}{\sqrt{t+3}+2} = \frac{1}{4}.$$

10b Slope of the tangent line is $f'(1) = \frac{1}{4}$, so by the point-slope formula we have

$$y - 2 = \frac{1}{4}(x-1) \quad\Rightarrow\quad y = \frac{1}{4}x + \frac{7}{4}.$$

11 We have

$$f'(x) = \lim_{t\to x} \frac{f(t)-f(x)}{t-x} = \lim_{t\to x} \frac{\dfrac{1}{t+2} - \dfrac{1}{x+2}}{t-x} = \lim_{t\to x} \frac{-(t-x)}{(t-x)(t+2)(x+2)}$$

$$= \lim_{t\to x} \frac{-1}{(t+2)(x+2)} = -\frac{1}{(x+2)^2}.$$