Math 140 Exam #3 Key (Summer 2018)

1 We have $f'(x) = 4x^2 + 10x - 6 = 2(2x-1)(x+3)$, so $f'(x) = 0$ when $x = -3, \frac{1}{2}$. These are the critical points. We evaluate:

$$f(-4) = 18\tfrac{2}{3}, \quad f(-3) = 27, \quad f(\tfrac{1}{2}) = -\tfrac{19}{12}, \quad f(1) = \tfrac{1}{3}.$$

The absolute minimum value of f on $[-4, 1]$ is $f(\frac{1}{2}) = -\frac{19}{12}$, and the absolute maximum value is $f(-3) = 27$.

2a Domain is $\{x : x \neq \pm 1\}$. The only intercept is $(0, 0)$.

2b Vertical asymptotes: $x = \pm 1$. Horizontal asymptote: $y = 0$.

2c From

$$f'(x) = -\frac{3(x^2+1)}{(x^2-1)^2}$$

we find there are no critical points.

2d We see in #2c that $f'(x) < 0$ for all $x \neq \pm 1$, and so f is decreasing on $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$. There are no local extrema.

2e We have

$$f''(x) = \frac{6x(x^2+3)}{(x^2-1)^3}$$

so $f''(x) > 0$ for $x \in (1, \infty)$, and $f''(x) < 0$ for $x \in (-\infty, -1) \cup (-1, 1)$. Therefore f is concave up on $(1, \infty)$, and concave down on $(-\infty, -1)$ and $(-1, 1)$. There is no inflection point at $x = 1$ since $f(1)$ is undefined.

3 Let x and y be the lengths of the sides of the right triangle whose hypotenuse is the crease, and let z be the distance point P is above the bottom of the paper. In the figure on the next page, at right, there are two right triangles. One of them, by the Pythagorean Theorem, gives

$$z^2 + (w-x)^2 = x^2 \quad \Rightarrow \quad z = \sqrt{2wx - w^2}.$$

The other triangle gives

$$w^2 + (y-z)^2 = y^2 \quad \Rightarrow \quad 2yz = w^2 + z^2 \quad \Rightarrow \quad y = \frac{wx}{\sqrt{2wx - w^2}}$$

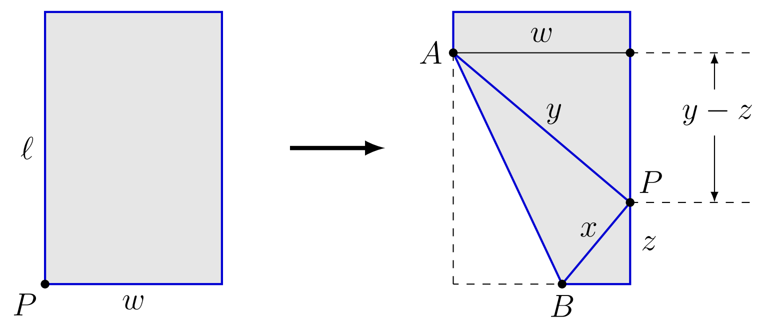
The square of the length $L(x)$ of the crease is thus

$$L^2(x) = x^2 + y^2 = \frac{2x^3}{2x - w},$$

We need to minimize $L^2(x)$ on the interval $x \in (w/2, w]$, so in particular $x \neq 0$. (If $x \leq w/2$ a crease cannot be made in the prescribed manner.) We have

$$(L^2)'(x) = \frac{2x^2(4x-3w)}{(2x-w)^2},$$

so $(L^2)'(x) = 0$ iff $x = 3w/4$. Since $(L^2)'(x) < 0$ for $x < 3w/4$ and $(L^2)'(x) > 0$ for $x > 3w/4$, the First Derivative Test implies L^2 has a minimum at $x = 3w/4$. Shortest crease length is therefore $L(3w/4) = \frac{3\sqrt{3}}{4}w$, which occurs when P is $z = \sqrt{2w(3w/4) - w^2} = w/\sqrt{2}$ above the bottom of the page. (Note that the value ℓ does not figure into any of this; however, it's important to have $\ell > w$ so that making creases is even possible.)



4 We have $S(r) = \pi r\sqrt{r^2+36}$. Since

$$S'(r) = \pi\sqrt{r^2+36} + \frac{\pi r}{\sqrt{r^2+36}},$$

the slope of the tangent line to the graph of $S(r)$ at $(10, S(10)) = (10, 20\pi\sqrt{34})$ is

$$S'(10) = 2\pi\sqrt{34} + \frac{5\pi}{\sqrt{34}},$$

and so the tangent line is

$$L(r) = \left(2\pi\sqrt{34} + \frac{5\pi}{\sqrt{34}}\right)r - \frac{50\pi}{\sqrt{34}} = \frac{73\pi}{\sqrt{34}}r - \frac{50\pi}{\sqrt{34}}$$

For r "near" 10 we have $S(r) \approx L(r)$, so the change in surface area is approximately

$$S(9.9) - S(10) \approx L(9.9) - L(10) = \left[\frac{73\pi}{\sqrt{34}}(9.9) - \frac{50\pi}{\sqrt{34}}\right] - \left[\frac{73\pi}{\sqrt{34}}(10) - \frac{50\pi}{\sqrt{34}}\right] = -\frac{73\pi}{10\sqrt{34}}.$$

5 By the Mean Value Theorem there exists some $c \in (-2, 14)$ such that

$$f'(c) = \frac{f(14) - f(-2)}{14 - (-2)} = \frac{7 - f(-2)}{16}.$$

Since $f'(c) \leq 10$ is required, we must have $7 - f(-2) \leq 160$, or $f(-2) \geq -153$.

6a 0/0 form. Using L'Hôpital's Rule twice:

$$\lim_{x\to 0} \frac{\sin^2 3x}{x^2} = \lim_{x\to 0} \frac{6\sin 3x \cos 3x}{2x} = \lim_{x\to 0} \frac{18\cos^2 3x - 18\sin^2 3x}{2} = 9.$$

6b ∞/∞ form. L'Hôpital's Rule gives

$$\lim_{x\to \pi/2} \frac{2\tan x}{\sec^2 x} = \lim_{x\to \pi/2} \frac{2\sec^2 x}{2\sec^2 x \tan x} = \lim_{x\to \pi/2} \frac{\cos x}{\sin x} = \frac{\cos(\pi/2)}{\sin(\pi/2)} = 0.$$

7a $\displaystyle\int \left(\frac{7}{t^4} + 8\sqrt{t}\right) dt = -\frac{7}{3t^3} + \frac{16}{3} t^{3/2} + C.$

7b $\displaystyle\int (5z^4 - 16\sec^2 2z)\, dz = z^5 - 8\tan 2z + C.$