Math 140 Exam #1 Key (Summer 2018)

1 Let $\epsilon > 0$. Choose $\delta = \epsilon/4$. Suppose $x \in \mathbb{R}$ is such that $0 < |x+3| < \delta$. Then $|x+3| < \epsilon/4$, and since

$$|x+3| < \frac{\epsilon}{4} \quad\Rightarrow\quad 4|x+3| < \epsilon \quad\Rightarrow\quad |4x+12| < \epsilon \quad\Rightarrow\quad |(4x+10)-(-2)| < \epsilon,$$

we conclude that $4x + 10 \to -2$ as $x \to -3$.

2a Since $\lim_{x\to-5^-} f(x) = \lim_{x\to-5^-}(1) = 1$ and

$$\lim_{x\to-5^+} f(x) = \lim_{x\to-5^+} \sqrt{25-x^2} = \sqrt{25-(-5)^2} = 0,$$

we see $\lim_{x\to-5} f(x)$ does not exist.

2b $\lim_{x\to5^-} f(x) = \lim_{x\to5^-} \sqrt{25-x^2} = \sqrt{25-5^2} = 0.$

2c $\lim_{x\to5^+} f(x) = \lim_{x\to5^+}(2x-10) = 0.$

2d From above, $\lim_{x\to5^-} f(x) = 0 = \lim_{x\to5^+} f(x)$, and so $\lim_{x\to5} f(x) = 0.$

2e $\lim_{x\to3} f(x) = \lim_{x\to3} \sqrt{25-x^2} = \sqrt{25-3^2} = 4.$

3a Factoring,

$$\lim_{t\to2} \frac{3t^2-7t+2}{2-t} = \lim_{t\to2} \frac{(3t-1)(t-2)}{2-t} = \lim_{t\to2}(1-3t) = 1-3(2) = -5.$$

3b Factoring again,

$$\lim_{x\to49} \frac{\sqrt{x}-7}{x-49} = \lim_{x\to49} \frac{\sqrt{x}-7}{(\sqrt{x}-7)(\sqrt{x}+7)} = \lim_{x\to49} \frac{1}{\sqrt{x}+7} = \frac{1}{\sqrt{49}+7} = \frac{1}{14}.$$

4 Since $\lim_{x\to2^-} h(x) = \lim_{x\to2^-}(b-3x) = b-6$ and $\lim_{x\to2^+} h(x) = \lim_{x\to2^+}(x+2) = 4$, we need $b-6=4$, or $b=10$. Then the two-sided limit exists and has value 4.

5 Since

$$f(x) = \frac{(x-7)(x-2)}{(x-3)(x-2)} = \frac{x-7}{x-3},$$

f has a vertical asymptote at $x = 3$. Indeed, we find that

$$\lim_{x\to3^+} f(x) = -\infty \quad\text{and}\quad \lim_{x\to3^-} f(x) = +\infty.$$

All we can say about $\lim_{x\to3} f(x)$ is that it does not exist.

6 Divide by the highest power of x in the denominator:

$$\lim_{x\to\infty}\frac{7-4x^2}{6x^2+5x+2}\cdot\frac{x^{-2}}{x^{-2}}=\lim_{x\to\infty}\frac{7x^{-2}-4}{6+5x^{-1}+2x^{-2}}=\frac{-4}{6}=-\frac{2}{3}.$$

7 Recall that $\sqrt{x^2}=|x|$, so $\sqrt{x^2}=x$ if $x\geq 0$ and $\sqrt{x^2}=-x$ if $x<0$. Now, since $x\to\infty$ implies $x>0$,

$$\lim_{x\to\infty}\frac{\sqrt{x^2+1}}{2x+1}=\lim_{x\to\infty}\frac{\sqrt{x^2(1+x^{-2})}}{2x+1}=\lim_{x\to\infty}\frac{x\sqrt{1+x^{-2}}}{2x+1}=\lim_{x\to\infty}\frac{\sqrt{1+x^{-2}}}{2+x^{-1}}=\frac{\sqrt{1+0}}{2+0}=\frac{1}{2};$$

and since $x\to-\infty$ implies $x<0$,

$$\lim_{x\to-\infty}\frac{\sqrt{x^2+1}}{2x+1}=\lim_{x\to-\infty}\frac{\sqrt{x^2(1+x^{-2})}}{2x+1}=\lim_{x\to\infty}\frac{-x\sqrt{1+x^{-2}}}{2x+1}=\lim_{x\to\infty}\frac{-\sqrt{1+x^{-2}}}{2+x^{-1}}=-\frac{1}{2}.$$

Horizontal asymptotes are $y=\frac{1}{2}$ and $y=-\frac{1}{2}$.

8 Since

$$\lim_{x\to4^-}f(x)=\lim_{x\to4^-}(x^2-5)=4^2-5=11\neq 13=f(4),$$

f is not continuous at 4.

9 Continuity from the left at 1 requires that $\lim_{x\to1^-}g(x)=g(1)$. Since

$$\lim_{x\to1^-}g(x)=\lim_{x\to1^-}(x^2+x)=1^2+1=2$$

and $g(1)=a$, we set $a=2$ to secure continuity from the left at 1.

Continuity from the right at 1 requires that $\lim_{x\to1^+}g(x)=g(1)$. Since

$$\lim_{x\to1^+}g(x)=\lim_{x\to1^+}(3x+5)=3(1)+5=8$$

and $g(1)=a$, we set $a=8$ to secure continuity from the right at 1.

We see that there can be no value for a that results in continuity from the left and right at 1 simultaneously, which means there is no a value which will make g continuous at 1.

10a By definition,

$$\begin{aligned}f'(4)&=\lim_{h\to0}\frac{f(4+h)-f(4)}{h}=\lim_{h\to0}\frac{\sqrt{4+h}-2}{h}\\&=\lim_{h\to0}\frac{(\sqrt{4+h}-2)(\sqrt{4+h}+2)}{h(\sqrt{4+h}+2)}=\lim_{h\to0}\frac{h}{h(\sqrt{4+h}+2)}\\&=\lim_{h\to0}\frac{1}{\sqrt{4+h}+2}=\frac{1}{\sqrt{4+0}+2}=\frac{1}{4}.\end{aligned}$$

10b Slope of the tangent line is $f'(4)=\frac{1}{4}$, so by the point-slope formula we have

$$y-2=\tfrac{1}{4}(x-4)\quad\Rightarrow\quad y=\tfrac{1}{4}x+1$$

is the equation of the line.