Math 140 Exam #3 Key (Summer 2014)

1 Suppose that f is a continuous function such that $f(-3) = 11$ and $f(9) = -5$. Now, if f is differentiable on $(-3, 9)$, then by the Mean Value Theorem there exists some $c \in (-3, 9)$ such that

$$f'(c) = \frac{f(9) - f(-3)}{9 - (-3)} = \frac{-5 - 11}{12} = -\frac{4}{3} < -1.$$

That is, if f is differentiable on $(-3, 9)$ then it is not possible to have $f'(x) \geq -1$ for all $x \in (-3, 9)$. There can exist no function f of the kind proposed in the problem.

2 Let $f(x) = 3x - 1 - 2\cos x$, so f is everywhere continuous and differentiable. Since $f(0) = -3 < 0$ and $f(2\pi) = 6\pi - 3 > 0$, by the Intermediate Value Theorem there exists some $c \in (0, 2\pi)$ such that $f(c) = 0$, and thus $3c - 1 - 2\cos c = 0$. We now have established that the equation $3x - 1 - 2\cos x = 0$ has at least one real root.

Suppose that the equation has two real roots a and b, where $a < b$. Then $f(a) = f(b) = 0$, and so by Rolle's Theorem there exists some $c \in (a, b)$ such that $f'(c) = 0$. Since $f'(x) = 3 + 2\sin x$ this means that $3 + 2\sin c = 0$, which implies that

$$\sin c = -\tfrac{3}{2}$$

—impossible for any real number c! To avoid such a contradiction we must conclude that the equation $3x - 1 - 2\cos x = 0$ cannot have more than one real root, and therefore it must have exactly one real root.

3 Since $p'(t) = \frac{1}{2}t^{-1/2}$, we have

$$p(t) = t^{1/2} + c = \sqrt{t} + c$$

for some constant c. Now, $p(4) = 6$ implies that $6 = \sqrt{4} + c$, and thus $c = 4$. Therefore

$$p(t) = \sqrt{t} + 4.$$

4a Use the Power Rule:

$$\int \left(x^{3/4} + x^{5/2}\right) dx = \frac{4}{7}x^{7/4} + \frac{2}{7}x^{7/2} + c = \frac{4}{7}\sqrt[4]{x^7} + \frac{2}{7}\sqrt{x^7} + c$$
$$= \frac{4}{7}x\sqrt[4]{x^3} + \frac{2}{7}x^3\sqrt{x} + c.$$

(Note that the original indefinite integral only makes sense for $x > 0$.)

4b We have

$$\int \sec(5y)\tan(5y)dy = \frac{1}{5}\sec(5y) + c.$$

4c Let $u = 4 - 9x^2$, so by the formal mechanism of "u-substitution" we get

$$du = -18x\,dx \quad \Rightarrow \quad -\frac{1}{18}du = xdx,$$

and so

$$\int \frac{x}{\sqrt{4-9x^2}}dx = -\frac{1}{18}\int \frac{1}{\sqrt{u}}du = -\frac{1}{18}\int u^{-1/2}du = -\frac{1}{18}\cdot 2u^{1/2} = -\frac{1}{9}\sqrt{4-9x^2}+C.$$

5 We can partition $[1,5]$ into n subintervals each of length $\Delta x = \frac{5-1}{n} = \frac{4}{n}$, and evaluate $f(x) = 4x-3$ at the right endpoint of each subinterval so that $x_i^* = 1 + \frac{4}{n}i$. By definition,

$$\begin{aligned}\int_1^5 f(x)\,dx &= \lim_{n\to\infty}\sum_{i=1}^n f(x_i^*)\Delta x_i = \lim_{n\to\infty}\sum_{i=1}^n \left[4\left(1+\frac{4}{n}i\right)-3\right]\cdot\frac{4}{n} \\ &= \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^n\left(1+\frac{16}{n}i\right) = \lim_{n\to\infty}\frac{4}{n}\left(n+\frac{16}{n}\sum_{i=1}^n i\right) \\ &= \lim_{n\to\infty}\frac{4}{n}\left(n+\frac{16}{n}\cdot\frac{n(n+1)}{2}\right) = \lim_{n\to\infty}\left(36+\frac{32}{n}\right) = 36.\end{aligned}$$

6a $\int_6^2 7f = -7\int_2^6 f = -7(-4) = 28$

6b $\int_2^6 (f-3g) = \int_2^6 f - 3\int_2^6 g = -4-3(7) = -25$

6c We have

$$\int_2^5 9g = 9\int_2^5 g = 9\left(\int_2^6 g - \int_5^6 g\right) = 9(7-20) = 9(-13) = -117.$$

7 By the Fundamental Theorem of Calculus, and also the Chain Rule, we have

$$\Phi'(x) = \tan^2 x\cos^9(6-\tan x)\cdot(\tan x)' = \tan^2 x\sec^2 x\cos^9(6-\tan x)$$

8a We have

$$\int_1^9 \frac{3x^6-2\sqrt{x}}{x^2}dx = \int_1^9 (3t^4-2x^{-3/2})dt = \left[\frac{3}{5}x^5+\frac{4}{\sqrt{x}}\right]_1^9 = \frac{531,392}{15} = 35,426\tfrac{2}{15}.$$

8b Let $u = \cos\theta$, so $du = -\sin\theta\,d\theta$. When $x=0$ we get $u=1$ also; and when $x = \pi/4$ we get $u = 1/\sqrt{2}$. Thus we obtain

$$\int_0^{\pi/4}\cos^2\theta\sin\theta\,d\theta = -\int_1^{1/\sqrt{2}} u^2\,du = -\frac{1}{3}\left[u^3\right]_1^{1/\sqrt{2}} = -\frac{1}{3}\left[\left(\frac{1}{\sqrt{2}}\right)^3-1\right] = \frac{1}{3}\left(1-\frac{1}{2\sqrt{2}}\right).$$

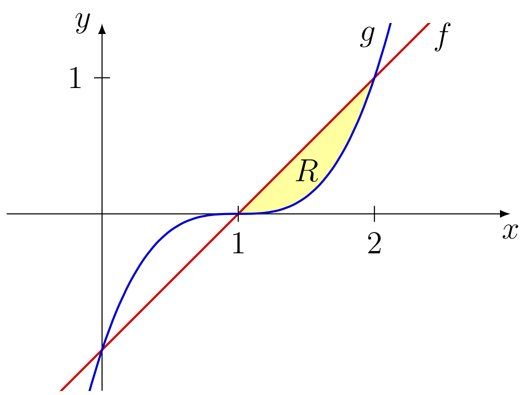
9 First we find the points where the curves generated by $f(x) = x - 1$ and $g(x) = (x-1)^3$ intersect:

$$f(x) = g(x) \quad \Rightarrow \quad x - 1 = (x-1)^3 \quad \Rightarrow \quad (x-1)^3 - (x-1) = 0$$
$$\Rightarrow \quad x(x-1)(x-2) = 0 \quad \Rightarrow \quad x = 0, 1, 2.$$

so the points are $(0, f(0)) = (0, -1)$, $(1, f(1)) = (1, 0)$ and $(2, f(2)) = (2, 1)$. The point $(0, -1)$ is in the fourth quadrant and so can be discarded. The region R enclosed by f and g between $x = 1$ and $x = 2$ is easily verified to lie in the first quadrant, with $f(x) \geq g(x)$ for $1 \leq x \leq 2$ in particular. Thus

$$\mathcal{A}(R) = \int_1^2 [f(x) - g(x)]\, dx = \int_1^2 [(x-1) - (x-1)^3]\, dx$$
$$= \int_1^2 (-x^3 + 3x^2 - 2x)\, dx = \left[-\frac{1}{4}x^4 + x^3 - x^2 \right]_1^2 = \frac{1}{4}.$$

See the figure below.



10 For each $x \in [0, 5]$ we find that the cross-sectional area is

$$A(x) = \left(2\sqrt{25 - x^2} \right)^2 = 4(25 - x^2),$$

and thus the volume of the solid is

$$V = \int_0^5 A(x)\, dx = 4 \int_0^5 (25 - x^2)\, dx = 4\left[25x - \frac{x^3}{3} \right]_0^5 = 4\left[25(5) - \frac{5^3}{3} \right] = \frac{1000}{3}.$$

11 The volume is

$$V = \int_0^{1/2} \pi \left(\frac{1}{\sqrt[4]{1-x}} \right)^2 dx = \pi \int_0^{1/2} (1-x)^{-1/2} dx = \pi \left[-2(1-x)^{1/2} \right]_0^{1/2} = \left(2 - \sqrt{2} \right) \pi.$$