Math 140 Exam #2 Key (Summer 2014)

1a By the Chain Rule,

$$y' = 16(8 - 6t)(8t - 3t^2)^{15}$$

1b Again by the Chain Rule,

$$y' = 6\sin^5\theta\cos\theta - 6\cos\theta\sin\theta.$$

1c Applying the Chain Rule twice,

$$f'(x) = \tfrac{1}{2}(x - x^{1/2})^{-1/2} \cdot (x - x^{1/2})' = \tfrac{1}{2}(x - x^{1/2})^{-1/2} \cdot (1 - \tfrac{1}{2}x^{-1/2}),$$

or

$$f'(x) = \frac{1 - \frac{1}{2}x^{-1/2}}{2\sqrt{x - \sqrt{x}}} = \frac{2\sqrt{x} - 1}{2\sqrt{x}\sqrt{x - \sqrt{x}}} = \frac{2\sqrt{x} - 1}{4\sqrt{x^2 - x\sqrt{x}}}.$$

2a Differentiating both sides with respect to x gives

$$4x^3 + 4xy^2 + 4x^2yy' + 4y^3y' = \frac{25}{4}y^2 + \frac{25}{2}xyy'.$$

Solving for y':

$$y' = \frac{25y^2 - 16xy^2 - 16x^3}{16x^2y + 16y^3 - 50xy}.$$

2b At $(x, y) = (1, 2)$ we have $y' = 1/3$, and so the equation of the tangent line is given by $y - 2 = \frac{1}{3}(x - 1)$, or

$$y = \frac{1}{3}x + \frac{5}{3}.$$

3 We differentiate the formula $A = \frac{1}{2}bh$ with respect to time t to obtain

$$\frac{dA}{dt} = \frac{1}{2}\left(b\frac{dh}{dt} + h\frac{db}{dt}\right). \qquad (1)$$

When area is $A = 150$ cm^2 and height is $h = 12$ cm we find the base to be $b = 2A/h = 25$ cm. Putting all known quantities into (1) gives

$$2 = \frac{1}{2}\left(25(-1) + 12\frac{db}{dt}\right),$$

or $db/dt = 2\frac{5}{12}$ cm/min.

4 The ground, wall, and ladder form a right triangle with hypotenuse of length 13. If x is the distance between the wall and the foot of the ladder, and y is the distance between the ground and the top of the ladder, then $x^2 + y^2 = 13^2$. Both x and y are functions of time t, and so differentiating with respect to t yields

$$2x(t)x'(t) + 2y(t)y'(t) = 0 \quad \Rightarrow \quad y'(t) = -\frac{x(t)x'(t)}{y(t)}.$$

We're given that $x'(t) = 0.5$ ft/s for all $t \geq 0$, and of course we also have $y(t) = \sqrt{169 - x^2(t)}$. Thus

$$y'(t) = -\frac{x(t)}{2\sqrt{169 - x^2(t)}}.$$

Now, at the time t when $x(t) = 5$ ft, we obtain

$$y'(t) = -\frac{5}{2\sqrt{169 - 5^2}} = -\frac{5}{24};$$

that is, at the time t when the foot of the ladder is 5 ft from the wall, the top of the ladder is sliding down the wall at a rate of $-5/24$ ft/s.

5 We have $f'(x) = 3x^2 - 4x - 5$. Setting $f'(x) = 0$ gives the quadratic equation $3x^2 - 4x - 5 = 0$, which has solutions

$$x = \frac{2 \pm \sqrt{19}}{3} \approx 2.12,\ -0.79.$$

Neither of these critical points lies in $[4, 8]$, so we need only evaluate f at the endpoints of the interval: $f(4) = 18$ is the global minimum and $f(8) = 350$ the global maximum.

6a $\text{Dom}(f) = \{x : x \neq \pm 1\} = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$. Intercepts are $(\frac{5}{3}, 0)$ and $(0, 5)$.

6b $y = 0$ is the horizontal asymptote since $f(x) = p(x)/q(s)$ with $\deg(p) = 1 < 2 = \deg(q)$. Also, since $f(x)$ is in reduced form, the vertical asymptotes are located at the zeros of its denominator: $x = -1$ and $x = 1$.

6c We find f':

$$f'(x) = \frac{(x^2-1)(3) - (3x-5)(2x)}{(x^2-1)^2} = \frac{(1-3x)(x-3)}{(x^2-1)^2}.$$

Now, with the understanding that $x \neq \pm 1$,

$$\begin{array}{llcccccc} f'(x) > 0 & \Rightarrow & 1-3x > 0 & \text{and} & x-3 > 0 & \text{or} & 1-3x < 0 & \text{and} & x-3 < 0 \\ & \Rightarrow & x < \frac{1}{3} & \text{and} & x > 3 & \text{or} & x > \frac{1}{3} & \text{and} & x < 3 \\ & \Rightarrow & & \varnothing & & \text{or} & & x \in \left(\frac{1}{3}, 3\right) & \end{array}$$

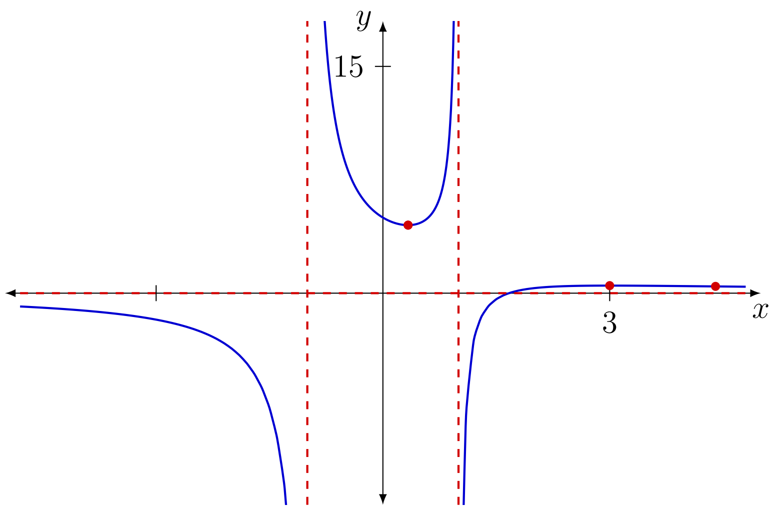
Hence $f' > 0$ on $(1/3, 1)$ and $(1, 3)$, and so f is increasing on these intervals by the Monotonicity Test. By similar reasoning we conclude that $f' < 0$ on $(-\infty, -1)$, $(-1, 1/3)$, and $(3, \infty)$, so f is decreasing on these intervals. The only critical points of f are at $x = 1/3$ and $x = 3$. By the First Derivative Test we find that $\left(\frac{1}{3}, f(\frac{1}{3})\right) = \left(\frac{1}{3}, \frac{9}{2}\right)$ is a local minimum for f, and $(3, f(3)) = \left(3, \frac{1}{2}\right)$ is a local maximum for f.

6d The second derivative of f is

$$f''(x) = \frac{2(3x^3 - 15x^2 + 9x - 5)}{(x^2 - 1)^3}$$

That is, $f''(x) = p(x)/q(x)$, with $p(x) = 6x^3 - 30x^2 + 18x - 10$ and $q(x) = (x^2 - 1)^3$. With a calculator it can be determined that, to the nearest hundredth, $q(x) < 0$ on $(-\infty, 4.40)$ and $q(x) > 0$ on $(4.40, \infty)$. Also $p(x) < 0$ on $(-1, 1)$ and $p(x) > 0$ on $(-\infty, -1) \cup (1, \infty)$. Hence $f'' < 0$ on $(-\infty, -1) \cup (1, 4.40)$ and $f'' > 0$ on $(-1, 1) \cup (4.40, \infty)$. Therefore f is concave down on $(-\infty, -1) \cup (1, 4.40)$ and concave up on $(-1, 1) \cup (4.40, \infty)$. There is an inflection point at about $(4.40, 0.45)$.

6e Local extrema and inflection point are marked in red.



7 Let x be the length of the dividing fence, so that the outer rectangle has dimensions x and $216/x$. The length of fence ℓ used is

$$\ell(x) = 3x + \frac{432}{x}.$$

Differentiating:

$$\ell'(x) = 3 - \frac{432}{x^2}.$$

Set $\ell'(x) = 0$ to get the equation $3x^2 - 432 = 0$, and then $x^2 - 144 = 0$. The only physically meaningful solution is $x = 12$ m, which will be the value that minimizes $\ell(x)$. Amount of fence needed:

$$\ell(12) = 3(12) + \frac{432}{12} = 72 \text{ meters.}$$

8 Let x be as in the figure below (i.e. x is the distance between the nearest point on the shore to the island and the point where the cable will meet the shore). Cost function is:

$$C(x) = \left(\frac{\$2400}{\text{km}}\right)\left(\sqrt{x^2+3.5^2}\text{ km}\right) + \left(\frac{\$1200}{\text{km}}\right)(8-x\text{ km}) = 2400\sqrt{x^2+\frac{49}{4}} + 1200(8-x).$$

We take the derivative:

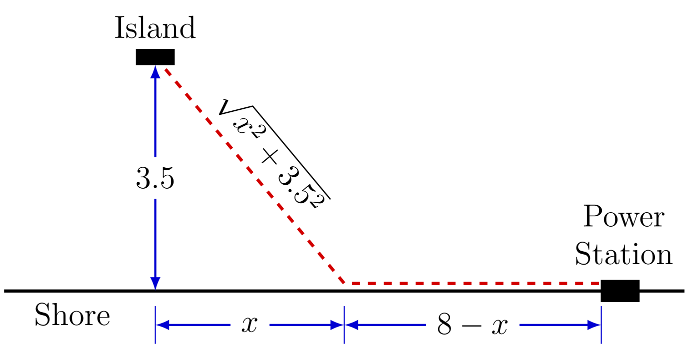
$$C'(x) = \frac{2400x}{\sqrt{x^2+49/4}} - 1200.$$

Note that there is no x value for which $C'(x)$ does not exist. On the other hand,

$$C'(x) = 0 \quad\Rightarrow\quad \frac{2x}{\sqrt{x^2+49/4}} - 1 = 0 \quad\Rightarrow\quad 2x = \sqrt{x^2+\frac{49}{4}} \quad\Rightarrow\quad 4x^2 = x^2 + \frac{49}{4}$$

$$\Rightarrow\quad x^2 = \frac{49}{12} \quad\Rightarrow\quad x = \sqrt{\frac{49}{12}} = \frac{7}{2\sqrt{3}} = \frac{7\sqrt{3}}{6} \approx 5.98.$$

Thus, if the cable meets the shore at a point about $8 - 5.98 = 2.02$ km to the left of the power station, cost will be minimized.



9a From $f'(x) = \frac{1}{5}x^{-4/5}$ we obtain $f'(32) = \frac{1}{5}(32)^{-4/5} = \frac{1}{80}$, which is the slope of the line L. A point on L is $(32, f(32)) = (32, 2)$. By the point-slope formula L has equation

$$y = \frac{1}{80}x + \frac{8}{5}.$$

9b Using

$$L(x) = \frac{1}{80}x + \frac{8}{5}$$

as a linear approximation of f, we have

$$\sqrt[5]{33} = f(33) \approx L(33) = \frac{1}{80}(33) + \frac{8}{5} = 2.0125.$$

(Note: this is quite close to the actual value 2.01234661...)