Math 140 Exam #1 Key (Summer 2014)

1a $\lim_{x\to 1^-} h(x) = 2$

1b $\lim_{x\to 1^+} h(x) = 4$

1c $\lim_{x\to 1} h(x) = \text{DNE}$

1d $\lim_{x\to 3^-} h(x) = 1$

1e $\lim_{x\to -2} h(x) = 1$

2a $(r^4 - 7r + 4)^{2/3}$ is a composition of a polynomial function with a radical function, and 3 is in its domain. Therefore, by direct substitution,

$$\lim_{r\to 3}(r^4 - 7r + 4)^{2/3} = [(3)^4 - 7(3) + 4]^{2/3} = \left(\sqrt[3]{64}\right)^2 = 4^2 = 16.$$

2b Combine the fractions for best results:

$$\lim_{t\to -2}\left(\frac{t^2}{t+2} + \frac{2t}{t+2}\right) = \lim_{t\to -2}\frac{t^2+2t}{t+2} = \lim_{t\to -2}\frac{t(t+2)}{t+2} = \lim_{t\to -2} t = -2.$$

2c Multiply by conjugate of the numerator:

$$\lim_{x\to 0}\left(\frac{\sqrt{2x^2+25}-5}{x^2}\cdot\frac{\sqrt{2x^2+25}+5}{\sqrt{2x^2+25}+5}\right) = \lim_{x\to 0}\frac{(2x^2+25)-25}{x^2(\sqrt{2x^2+25}+5)}$$

$$= \lim_{x\to 0}\frac{2x^2}{x^2(\sqrt{2x^2+25}+5)} = \lim_{x\to 0}\frac{2}{\sqrt{2x^2+25}+5} = \frac{2}{\sqrt{25}+5} = \frac{1}{5}.$$

3 Let

$$f(x) = x^2 - 5x - 2\cos x \quad \text{and} \quad h(x) = \sin x - 2.$$

Since $\lim_{x\to 0^+} f(x) = -2$ and $\lim_{x\to 0^+} h(x) = -2$, by the Squeeze Theorem it follows that $\lim_{x\to 0^+} g(x) = -2$ also.

4 Recall that in general $\sqrt{x^2} = |x|$. Now, when $x\to\infty$ we have $x > 0$, so then $\sqrt{x^2} = x$ and we obtain

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{4x^3}{2x^3+\sqrt{9x^6+15x^4}} = \lim_{x\to\infty}\frac{4x^3}{2x^3+\sqrt{x^6(9+15/x^2)}}$$

$$= \lim_{x\to\infty}\frac{4x^3}{2x^3+|x|^3\sqrt{9+15/x^2}} = \lim_{x\to\infty}\frac{4x^3}{2x^3+x^3\sqrt{9+15/x^2}}$$

$$= \lim_{x\to\infty}\frac{4}{2+\sqrt{9+15/x^2}} = \frac{4}{2+\sqrt{9+0}} = \frac{4}{5}.$$

On the other hand $x\to -\infty$ implies $x < 0$, so then $\sqrt{x^2} = -x$ and we obtain

$$\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty}\frac{4x^3}{2x^3+\sqrt{9x^6+15x^4}} = \lim_{x\to -\infty}\frac{4x^3}{2x^3+\sqrt{x^6(9+15/x^2)}}$$

$$= \lim_{x\to-\infty} \frac{4x^3}{2x^3 + |x|^3\sqrt{9+15/x^2}} = \lim_{x\to-\infty} \frac{4x^3}{2x^3 - x^3\sqrt{9+15/x^2}}$$
$$= \lim_{x\to-\infty} \frac{4}{2 - \sqrt{9+15/x^2}} = \frac{4}{2-\sqrt{9+0}} = -4.$$

Hence the horizontal asymptotes of f are $y = \frac{4}{5}$ and $y = -4$.

5 The function f is continuous at 4 if and only if $\lim_{x\to 4} f(x) = f(4) = 13$; but

$$\lim_{x\to 4} f(x) = \lim_{x\to 4}(x^2 - 5) = 4^2 - 5 = 11 \neq 13 = f(4),$$

and therefore f is not continuous at 4.

6 Continuity from the left at 1 requires that $\lim_{x\to 1^-} g(x) = g(1)$. Since

$$\lim_{x\to 1^-} g(x) = \lim_{x\to 1^-}(x^2 + x) = 1^2 + 1 = 2$$

and $g(1) = a$, we set $a = 2$ to secure continuity from the left at 1.

Continuity from the right at 1 requires that $\lim_{x\to 1^+} g(x) = g(1)$. Since

$$\lim_{x\to 1^+} g(x) = \lim_{x\to 1^+}(3x + 5) = 3(1) + 5 = 8$$

and $g(1) = a$, we set $a = 8$ to secure continuity from the right at 1.

We see that there can be no value for a that results in continuity from the left and right at 1 simultaneously, which means there is no a value which will make g continuous at 1.

7a By the definition of derivative:

$$f'(x) = \lim_{t\to x} \frac{f(t) - f(x)}{t - x} = \lim_{t\to x} \frac{\sqrt{3t+1} - \sqrt{3x+1}}{t - x}$$
$$= \lim_{t\to x}\left(\frac{\sqrt{3t+1} - \sqrt{3x+1}}{t-x} \cdot \frac{\sqrt{3t+1} + \sqrt{3x+1}}{\sqrt{3t+1} + \sqrt{3x+1}}\right)$$
$$= \lim_{t\to x} \frac{(3t+1) - (3x+1)}{(t-x)(\sqrt{3t+1} + \sqrt{3x+1})} = \lim_{t\to x} \frac{3(t-x)}{(t-x)(\sqrt{3t+1} + \sqrt{3x+1})}$$
$$= \lim_{t\to x} \frac{3}{\sqrt{3t+1} + \sqrt{3x+1}} = \frac{3}{\sqrt{3x+1} + \sqrt{3x+1}} = \frac{3}{2\sqrt{3x+1}},$$

and so we have

$$f'(1) = \frac{3}{2\sqrt{3(1)+1}} = \frac{3}{4}.$$

7b From Problem 7a, the slope of tangent line is $f'(1) = 3/4$. Since the line contains the point $(1, 2)$, we have

$$y - 2 = \frac{3}{4}(x - 1),$$

or simply

$$y = \frac{3}{4}x + \frac{5}{4}.$$

8a By the Product Rule,

$$f'(x) = (20x^3 + 6x)(x^3 + 7) + (5x^4 + 3x^2 + 1)(3x^2).$$

8b By the Quotient Rule,

$$g'(t) = \frac{(t^2+1)(2t) - (t^2-1)(2t)}{(t^2+1)^2} = \frac{4t}{(t^2+1)^2}.$$

8c Product Rule again:

$$y' = (\sin x)(\sec^2 x) + (\cos x)(\tan x) = \sec x \tan x + \sin x.$$

8d Quotient Rule again:

$$y' = \frac{(1+\sin x)(2\cos x)' - (2\cos x)(1+\sin x)'}{(1+\sin x)^2} = \frac{(1+\sin x)(-2\sin x) - (2\cos x)(\cos x)}{(1+\sin x)^2},$$

or simply

$$y' = -\frac{2}{\sin x + 1}.$$

9a Velocity function: $v(t) = s'(t) = 6t^2 - 42t$. Setting $v(t) = 0$ gives

$$6t^2 - 42t = 0 \quad \Rightarrow \quad 6t(t-7) = 0 \quad \Rightarrow \quad t = 0, 7.$$

However $t = 7$ is outside the designated domain $[0, 6]$, so the object is at rest only at time $t = 0$.

9b Acceleration function: $a(t) = v'(t) = s''(t) = 12t - 42$. Setting $a(t) = 0$ gives $t = 7/2$, the time when the acceleration is zero. Also $a(t) < 0$ for $t \in [0, 7/2)$ and $a(t) > 0$ for $t \in (7/2, 6]$.