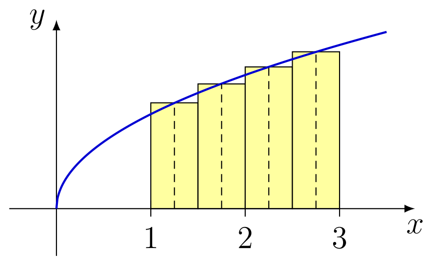
Math 140 Exam #4 Key (Summer 2013)

1a



1b By definition

$$\Delta x = \frac{3-1}{4} = \frac{1}{2}.$$

Also $x_0 = 1$, $x_1 = 1.5$, $x_2 = 2$, $x_3 = 2.5$, $x_4 = 3$, and $\bar{x}_0 = 1.25$, $\bar{x}_1 = 1.75$, $\bar{x}_2 = 2.25$, $\bar{x}_3 = 2.75$.

1c We have

$$\sum_{i=1}^{4} f(\bar{x}_i)\Delta x = \sum_{i=1}^{4} \frac{1}{2}\sqrt{\bar{x}_i} = \frac{1}{2}\left(\sqrt{5/4} + \sqrt{7/4} + \sqrt{9/4} + \sqrt{11/4}\right) \approx 2.7996.$$

2 We can partition $[1,5]$ into n subintervals each of length $\Delta x = \frac{5-1}{n} = \frac{4}{n}$, and evaluate $f(x) = 4x - 3$ at the right endpoint of each subinterval so that $x_i^* = 1 + \frac{4}{n}i$. By definition,

$$\begin{aligned}
\int_1^5 f(x)\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i = \lim_{n\to\infty} \sum_{i=1}^{n} \left[4\left(1 + \frac{4}{n}i\right) - 3\right] \cdot \frac{4}{n} \\
&= \lim_{n\to\infty} \frac{4}{n} \sum_{i=1}^{n} \left(1 + \frac{16}{n}i\right) = \lim_{n\to\infty} \frac{4}{n}\left(n + \frac{16}{n}\sum_{i=1}^{n} i\right) \\
&= \lim_{n\to\infty} \frac{4}{n}\left(n + \frac{16}{n} \cdot \frac{n(n+1)}{2}\right) = \lim_{n\to\infty}\left(36 + \frac{32}{n}\right) = 36.
\end{aligned}$$

3a $\int_6^2 7f = -7\int_2^6 f = 7(-2) = -14$

3b $\int_2^6 (f - 3g) = \int_2^6 f - 3\int_2^6 g = 2 - 3(8) = -22$

3c We have

$$\int_2^5 9g = 9\int_2^5 g = 9\left(\int_2^6 g - \int_5^6 g\right) = 9[8 - (-4)] = 9(12) = 108.$$

4 By the Fundamental Theorem of Calculus, and also the Chain Rule, we have

$$\Phi'(x) = \cos x \sin^3(7\cos x) \cdot (\cos x)' = -\cos(x)\sin(x)\sin^3(7\cos x).$$

5a $$\int_1^4 \frac{5t^6 - \sqrt{t}}{t^2} dt = \int_1^4 (5t^4 - t^{-3/2}) dt = \left[t^5 + 2t^{-1/2}\right]_1^4 = (1024 + 1) - (1 + 2) = 1022.$$

5b Let $u = \sin\theta$, so $du = \cos\theta\, d\theta$. When $x = 0$ we get $u = 0$ also; and when $x = \pi/2$ we get $u = 1$. Thus we obtain

$$\int_0^{\pi/2} \sin^2\theta \cos\theta\, d\theta = \int_0^1 u^2\, du = \left[\frac{1}{3}u^3\right]_0^1 = \frac{1}{3}.$$

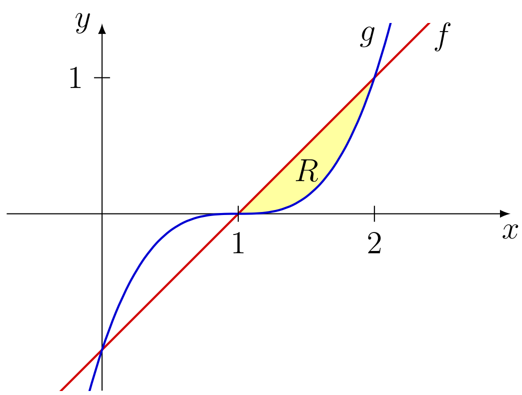
6 First we find the points where the curves generated by $f(x) = x - 1$ and $g(x) = (x-1)^3$ intersect:

$$\begin{aligned} f(x) = g(x) \;&\Rightarrow\; x - 1 = (x-1)^3 \;\Rightarrow\; (x-1)^3 - (x-1) = 0 \\ &\Rightarrow\; x(x-1)(x-2) = 0 \;\Rightarrow\; x = 0, 1, 2. \end{aligned}$$

so the points are $(0, f(0)) = (0, -1)$, $(1, f(1)) = (1, 0)$ and $(2, f(2)) = (2, 1)$. The point $(0, -1)$ is in the fourth quadrant and so can be discarded. The region R enclosed by f and g between $x = 1$ and $x = 2$ is easily verified to lie in the first quadrant, with $f(x) \geq g(x)$ for $1 \leq x \leq 2$ in particular. Thus

$$\begin{aligned} \mathcal{A}(R) &= \int_1^2 [f(x) - g(x)]\, dx = \int_1^2 [(x-1) - (x-1)^3]\, dx \\ &= \int_1^2 (-x^3 + 3x^2 - 2x)\, dx = \left[-\frac{1}{4}x^4 + x^3 - x^2\right]_1^2 = \frac{1}{4}. \end{aligned}$$

See the figure below.



7 Let $f(x) = x$ and $g(x) = x^n$. For any $n \geq 2$ we have $f \geq g$ on the interval $[0, 1]$, with $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$. This is where a region is enclosed by the two curves.

Area of the region is

$$\mathcal{A} = \int_0^1 (f - g) = \int_0^1 (x - x^n)dx = \left[\frac{x^2}{2} - \frac{x^{n+1}}{n+1}\right]_0^1 = \frac{1}{2} - \frac{1}{n+1}.$$

8 Let $f(x) = x$ and $g(x) = 2\sqrt{x}$. Now,

$$f(x) = g(x) \quad \Rightarrow \quad x = 2\sqrt{x} \quad \Rightarrow \quad x^2 = 4x \quad \Rightarrow \quad x(x-4) = 0 \quad \Rightarrow \quad x = 0, 4,$$

with $g \geq f$ on the interval $[0, 4]$. This interval is where the two curves enclose a region $\mathcal{R}$ in the xy-plane. To find the volume $\mathcal{V}$ of the solid generated by revolving R about the x-axis will require the Washer Method. We have

$$\begin{aligned}\mathcal{V} &= \int_0^4 \pi(g^2 - f^2) = \pi \int_0^4 \left[\left(2\sqrt{x}\right)^2 - x^2\right]dx = \pi \int_0^4 (4x - x^2)dx \\ &= \pi\left[2x^2 - \frac{1}{3}x^3\right]_0^4 = \pi\left(32 - \frac{64}{3}\right) = \frac{32}{3}\pi.\end{aligned}$$