Math 140 Exam #1 Key (Summer 2013)

1a $\displaystyle\lim_{x\to 1^-} h(x) = 2$

1b $\displaystyle\lim_{x\to 1^+} h(x) = 4$

1c $\displaystyle\lim_{x\to 1} h(x) = \text{DNE}$

1d $\displaystyle\lim_{x\to -1} h(x) = 3$

1e $\displaystyle\lim_{x\to 2} h(x) = 2$

2a $(r^4 - 7r + 4)^{2/3}$ is a composition of a polynomial function with a radical function, and 3 is in its domain. Therefore, by direct substitution,

$$\lim_{r\to 3}(r^4 - 7r + 4)^{2/3} = [(3)^4 - 7(3) + 4]^{2/3} = \left(\sqrt[3]{64}\right)^2 = 4^2 = 16.$$

2b Combine the fractions for best results:

$$\lim_{t\to -2}\left(\frac{t^2}{t+2} + \frac{2t}{t+2}\right) = \lim_{t\to -2}\frac{t^2+2t}{t+2} = \lim_{t\to -2}\frac{t(t+2)}{t+2} = \lim_{t\to -2} t = -2.$$

2c Multiply by conjugate of the numerator:

$$\lim_{x\to 0}\frac{\sqrt{2x^2+25}-5}{x^2}\cdot\frac{\sqrt{2x^2+25}+5}{\sqrt{2x^2+25}+5} = \lim_{x\to 0}\frac{(2x^2+25)-25}{x^2(\sqrt{2x^2+25}+5)}$$

$$= \lim_{x\to 0}\frac{2x^2}{x^2(\sqrt{2x^2+25}+5)} = \lim_{x\to 0}\frac{2}{\sqrt{2x^2+25}+5} = \frac{2}{\sqrt{25}+5} = \frac{1}{5}.$$

3 Let $f(x) = 2x^2 - 5x + \cos x$ and $h(x) = \sin x + 1$. Since $\lim_{x\to 0^+} f(x) = 1$ and $\lim_{x\to 0^+} h(x) = 1$, by the Squeeze Theorem it follows that $\lim_{x\to 0^+} g(x) = 1$ also.

4 Factor numerator and denominator:

$$f(x) = \frac{x+1}{x(x-2)^2}.$$

The rational expression is seen to be in reduced form, and so vertical asymptotes for f are $x = 0$ and $x = 2$. We have

$$\lim_{x\to 2^+} f(x) = \infty \quad\text{and}\quad \lim_{x\to 2^-} f(x) = \infty,$$

and so $\lim_{x\to 2} f(x) = \infty$. Also

$$\lim_{x\to 0^+} f(x) = \infty \quad\text{and}\quad \lim_{x\to 0^-} f(x) = -\infty,$$

and so $\lim_{x\to 0} f(x)$ can only be said to not exist.

5 Recall that in general $\sqrt{x^2} = |x|$. Now, when $x \to \infty$ we have $x > 0$, so then $\sqrt{x^2} = x$ and we obtain

$$\begin{aligned}\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{4x^3}{2x^3 + \sqrt{9x^6 + 15x^4}} = \lim_{x\to\infty} \frac{4x^3}{2x^3 + \sqrt{x^6(9 + 15/x^2)}} \\ &= \lim_{x\to\infty} \frac{4x^3}{2x^3 + |x|^3\sqrt{9 + 15/x^2}} = \lim_{x\to\infty} \frac{4x^3}{2x^3 + x^3\sqrt{9 + 15/x^2}} \\ &= \lim_{x\to\infty} \frac{4}{2 + \sqrt{9 + 15/x^2}} = \frac{4}{2 + \sqrt{9 + 0}} = \frac{4}{5}.\end{aligned}$$

On the other hand $x \to -\infty$ implies $x < 0$, so then $\sqrt{x^2} = -x$ and we obtain

$$\begin{aligned}\lim_{x\to-\infty} f(x) &= \lim_{x\to-\infty} \frac{4x^3}{2x^3 + \sqrt{9x^6 + 15x^4}} = \lim_{x\to-\infty} \frac{4x^3}{2x^3 + \sqrt{x^6(9 + 15/x^2)}} \\ &= \lim_{x\to-\infty} \frac{4x^3}{2x^3 + |x|^3\sqrt{9 + 15/x^2}} = \lim_{x\to-\infty} \frac{4x^3}{2x^3 - x^3\sqrt{9 + 15/x^2}} \\ &= \lim_{x\to-\infty} \frac{4}{2 - \sqrt{9 + 15/x^2}} = \frac{4}{2 - \sqrt{9 + 0}} = -4.\end{aligned}$$

Hence the horizontal asymptotes of f are $y = \frac{4}{5}$ and $y = -4$.

6 The function f is continuous at 4 if and only if $\lim_{x\to 4} f(x) = f(4) = 13$; but

$$\lim_{x\to 4} f(x) = \lim_{x\to 4} (x^2 - 5) = 4^2 - 5 = 11 \neq 13 = f(4),$$

and therefore f is not continuous at 4.

7 Continuity from the left at 1 requires that $\lim_{x\to 1^-} g(x) = g(1)$. Since

$$\lim_{x\to 1^-} g(x) = \lim_{x\to 1^-} (x^2 + x) = 1^2 + 1 = 2$$

and $g(1) = a$, we set $a = 2$ to secure continuity from the left at 1.

Continuity from the right at 1 requires that $\lim_{x\to 1^+} g(x) = g(1)$. Since

$$\lim_{x\to 1^+} g(x) = \lim_{x\to 1^+} (3x + 5) = 3(1) + 5 = 8$$

and $g(1) = a$, we set $a = 8$ to secure continuity from the right at 1.

We see that there can be no value for a that results in continuity from the left and right at 1 simultaneously, which means there is no a value which will make g continuous at 1.

8a By the definition of derivative:

$$f'(9) = \lim_{h\to 0} \frac{f(9+h) - f(9)}{h} = \lim_{h\to 0} \frac{\dfrac{1}{\sqrt{9+h}} - \dfrac{1}{3}}{h}$$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{\dfrac{1}{\sqrt{9+h}} - \dfrac{1}{3}}{h} \cdot \frac{3\sqrt{9+h}}{3\sqrt{9+h}} = \lim_{h\to 0} \frac{3-\sqrt{9+h}}{3h\sqrt{9+h}} \\
&= \lim_{h\to 0} \frac{3-\sqrt{9+h}}{3h\sqrt{9+h}} \cdot \frac{3+\sqrt{9+h}}{3+\sqrt{9+h}} = \lim_{h\to 0} \frac{-h}{3h\sqrt{9+h}\cdot(3+\sqrt{9+h})} \\
&= \lim_{h\to 0} \frac{-1}{3\sqrt{9+h}\cdot(3+\sqrt{9+h})} = \frac{-1}{3\sqrt{9}\cdot(3+\sqrt{9})} = -\frac{1}{54}.
\end{aligned}
$$

8b Slope will be $f'(9) = -1/54$, so equation is $y = -\frac{1}{54}x + \frac{1}{2}$.

9a By the definition of derivative:

$$
\begin{aligned}
f'(x) &= \lim_{t\to x} \frac{f(t)-f(x)}{t-x} = \lim_{t\to x} \frac{\sqrt{3t+1}-\sqrt{3x+1}}{t-x} \\
&= \lim_{t\to x}\left(\frac{\sqrt{3t+1}-\sqrt{3x+1}}{t-x} \cdot \frac{\sqrt{3t+1}+\sqrt{3x+1}}{\sqrt{3t+1}+\sqrt{3x+1}} \right) \\
&= \lim_{t\to x} \frac{(3t+1)-(3x+1)}{(t-x)(\sqrt{3t+1}+\sqrt{3x+1}\,)} = \lim_{t\to x} \frac{3(t-x)}{(t-x)(\sqrt{3t+1}+\sqrt{3x+1}\,)} \\
&= \lim_{t\to x} \frac{3}{\sqrt{3t+1}+\sqrt{3x+1}} = \frac{3}{\sqrt{3x+1}+\sqrt{3x+1}} = \frac{3}{2\sqrt{3x+1}}.
\end{aligned}
$$

9b Slope of tangent line is

$$
f'(8) = \frac{3}{2\sqrt{3(8)+1}} = \frac{3}{10},
$$

so equation is

$$
y - 5 = \frac{3}{10}(x-8),
$$

or simply

$$
y = \frac{3}{10}x + \frac{13}{5}.
$$

Extra Credit: Let $\epsilon > 0$. Choose $\alpha = \sqrt[3]{5/\epsilon}$. Suppose that $x > \alpha$. Then we have $x > \sqrt[3]{5/\epsilon}$, and since

$$
x > \sqrt[3]{5/\epsilon} \quad \Leftrightarrow \quad x^3 > \frac{5}{\epsilon} \quad \Leftrightarrow \quad \frac{x^3}{5} > \frac{1}{\epsilon} \quad \Leftrightarrow \quad \frac{5}{x^3} < \epsilon,
$$

it follows that

$$
\left| \frac{5}{x^3} - 0 \right| < \epsilon
$$

and therefore $\lim_{x\to\infty} 5/x^3 = 0$.