Math 140 Exam #3 Key (Summer 2012)

1a. We know $\sqrt[5]{32} = 2$, so get the equation of the tangent line to the curve f at $(32, 2)$. First,

$$f'(x) = \frac{1}{5}x^{-4/5} \quad \Rightarrow \quad f'(32) = \frac{1}{5}(32)^{-4/5} = \frac{1}{5} \cdot \frac{1}{16} = \frac{1}{80},$$

so the slope of the line is $\frac{1}{80}$. Now, using point-slope formula, we get

$$y - 2 = \frac{1}{80}(x - 32) \quad \Rightarrow \quad L(x) = \frac{1}{80}x + \frac{8}{5}$$

as the linearization for f centered at $x = 32$.

1b. $\sqrt[5]{33} = f(33) \approx L(33) = \frac{1}{80}(33) + \frac{8}{5} = 2.0125$. (Actual value is 2.012346617..., which amounts to only a 0.0076% error.)

2. Let $f(x) = 2x - 1 - \sin x$, so $f(0) = -1 < 0$ and $f(\pi) = 2\pi - 1 > 0$. Since f is continuous on $[0, \pi]$ and differentiable on $(0, \pi)$, the Intermediate Value Theorem implies there exists some $r \in (0, \pi)$ such that $f(r) = 0$. Now suppose that f has two real zeros r_1 and r_2, with $r_1 < r_2$. Since $f(r_1) = 0 = f(r_2)$, f is continuous on $[r_1, r_2]$, and f is differentiable on (r_1, r_2), by Rolle's Theorem there exists some $c \in (r_1, r_2)$ such that $f'(c) = 0$. Since $f'(x) = 2 - \cos x$, we obtain $\cos c = 2$. But this is a contradiction, since the cosine function cannot equal 2 at any real number! Hence f cannot have *more than* one real zero, and since it's been shown to have *at least* one real zero, we conclude that it must have *exactly* one real zero. Therefore $2x - 1 - \sin x = 0$ has exactly one real root.

3a. $\displaystyle\int (3x^{-2} - 4x^2 + 1)dx = -3x^{-1} - \frac{4}{3}x^3 + x + C$

3b. $\displaystyle\int [\cos(4t) - \sin(t/4)]dt = \frac{1}{4}\sin(4t) + 4\cos(t/4) + C$

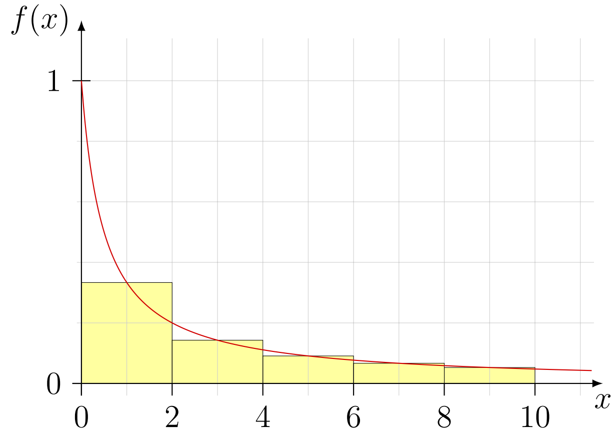
3c. Let $u = 1 - 4x^3$, so by the mechanism of "u-substitution" we get $du = -12x^2dx \Rightarrow -\frac{1}{6}du = 2x^2dx$, and so

$$\int \frac{2x^2}{\sqrt{1-4x^3}}dx = -\frac{1}{6}\int \frac{1}{\sqrt{u}}du = -\frac{1}{6}\int u^{-1/2}du = -\frac{1}{6} \cdot 2u^{1/2} = -\frac{1}{3}\sqrt{1-4x^3} + c,$$

where c is an arbitrary constant.

4. $\displaystyle f(x) = \int f'(x)dx = \int (8x - 5)dx = 4x^2 - 5x + c$, which implies that $f(0) = c$. But we're given $f(0) = 4$, so we obtain $c = 4$ and arrive at the solution $f(x) = 4x^2 - 5x + 4$.

5. Area $\approx 2[f(1)+f(3)+f(5)+f(7)+f(9)] = 2(1/3+1/7+1/11+1/15+1/19) = 10{,}042/7{,}315 \approx 1.3728$ m.



6. $\lim_{\Delta\to 0}\sum_{k=1}^{n} \bar{x}_k \cos \bar{x}_k \,\Delta x_k = \int_1^2 x\cos x\,dx$

7a. $\int_0^3 5f(x)\,dx = 5\int_0^3 f(x)\,dx = 5(2) = 10$

7b. $\int_3^6 [3f(x)-g(x)]dx = 3\int_3^6 f(x)\,dx - \int_3^6 g(x)\,dx = 3(-9)-5 = -32$

7c. $\int_6^3 [f(x)+2g(x)]dx = -\int_3^6 [f(x)+2g(x)]dx = -[(-9)+2(5)] = -1$

8. We can partition $[2,6]$ into n subintervals each of length $\Delta x = \frac{6-2}{n} = \frac{4}{n}$, and evaluate $f(x) = 3x^2-5$ at the right endpoint of each subinterval so that $x_i^* = 2+\frac{4}{n}i$. By definition,

$$\begin{aligned}
\int_2^6 f(x)\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x_i = \lim_{n\to\infty}\sum_{i=1}^{n}\left[3\left(2+\frac{4}{n}i\right)^2-5\right]\cdot\frac{4}{n} \\
&= \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n}\left(7+\frac{48}{n}i+\frac{48}{n^2}i^2\right) = \lim_{n\to\infty}\frac{4}{n}\left(7n+\frac{48}{n}\sum_{i=1}^{n}i+\frac{48}{n^2}\sum_{i=1}^{n}i^2\right) \\
&= \lim_{n\to\infty}\frac{4}{n}\left(7n+\frac{48}{n}\cdot\frac{n(n+1)}{2}+\frac{48}{n^2}\cdot\frac{n(n+1)(2n+1)}{6}\right) \\
&= \lim_{n\to\infty}\frac{188n^2+192n+32}{n^2} = 188.
\end{aligned}$$

9. $\dfrac{d}{dx}\displaystyle\int_7^{x^4} \sin^5(t)\,dt = \sin^5(x^4)\cdot 4x^3 = 4x^3\sin^5(x^4).$

10a. $\displaystyle\int_1^4 \frac{5t^6-\sqrt{t}}{t^2}dt = \int_1^4 (5t^4 - t^{-3/2})dt = \left[t^5 + 2t^{-1/2}\right]_1^4 = (1024+1)-(1+2) = 1022.$

10b. Let $u = \sin\theta$, so $du = \cos\theta\,d\theta$. When $x = 0$ we get $u = 0$ also; and when $x = \pi/2$ we get $u = 1$. Thus we obtain

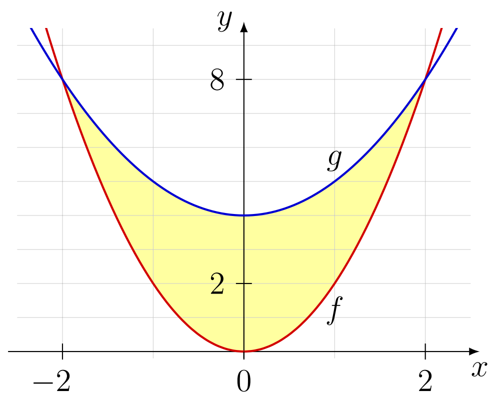
$$\int_0^{\pi/2} \sin^2\theta\cos\theta\,d\theta = \int_0^1 u^2\,du = \left[\frac{1}{3}u^3\right]_0^1 = \frac{1}{3}.$$

11. First we find the points where the curves generated by f and g intersect:

$$f(x) = g(x) \;\Rightarrow\; 2x^2 = x^2+4 \;\Rightarrow\; x^2 = 4 \;\Rightarrow\; x = \pm 2,$$

so the points are $(-2, f(-2)) = (-2, 8)$ and $(2, f(2)) = (2, 8)$. Between $x = -2$ and $x = 2$ we have $f(0) = 0 < 4 = g(0)$, so $g(x) \geq f(x)$ when $-2 \leq x \leq 2$. We now find the area A of the bounded region:

$$A = \int_{-2}^2 [g(x)-f(x)]dx = \int_{-2}^2 (4-x^2)dx = \left[4x - \frac{1}{3}x^3\right]_{-2}^2$$
$$= (8-8/3) - (-8+8/3) = 16 - 16/3 = \frac{32}{3}.$$



12. In the first quadrant the curve given by $f(x) = 4 - x^2$ goes from $(0, 4)$ to $(2, 0)$, so the limits of integration will be $x = 0$ and $x = 2$: Volume V is thus

$$V = \int_0^2 \pi[f(x)]^2\,dx = \pi\int_0^2 (4-x^2)^2\,dx = \pi\int_0^2 (16 - 8x^2 + x^4)\,dx$$
$$= \pi\left[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right]_0^2 = \pi\left[16(2) - \frac{8}{3}(8) + \frac{1}{5}(32)\right] = \frac{256}{15}\pi.$$