Math 140 Exam #2 Key (Summer 2012)

1. Using the general result $\lim_{x\to 0}(\sin ax)/x = a$, we have

$$\lim_{x\to 0}\frac{\tan 6x}{x} = \lim_{x\to 0}\frac{\sin 6x}{x}\cdot\frac{1}{\cos 6x} = \lim_{x\to 0}\frac{\sin 6x}{x}\cdot\lim_{x\to 0}\frac{1}{\cos 6x} = 6\cdot\frac{1}{\cos 0} = 6.$$

2. Quotient Rule:

$$f'(x) = \frac{(2-\tan x)(\sin x)' - (\sin x)(2-\tan x)'}{(2-\tan x)^2} = \frac{(2-\tan x)(\cos x) - (\sin x)(-\sec^2 x)}{(2-\tan x)^2}$$
$$= \frac{2\cos x - \sin x + \sin x\sec^2 x}{(2-\tan x)^2}$$

3. From $y'(x) = -\csc x\cot x$ we have $y'(\pi/4) = -\csc(\pi/4)\cot(\pi/4) = -\sqrt{2}$ as the slope of the tangent line. The equation is thus $y - \csc(\pi/4) = -\sqrt{2}(x-\pi/4)$, or

$$y = \sqrt{2}(\pi/4 - x) + \sqrt{2}.$$

4a. $f'(x) = 10(4x^3-9)^9\cdot 12x^2 = 120x^2(4x^3-9)^9.$

4b. $g'(t) = \cos(4\cos t)\cdot(-4\sin t) = -4\sin(t)\cos(4\cos t).$

4c. $h'(x) = \frac{1}{2}(x+x^{1/2})^{-1/2}\cdot(x+x^{1/2})' = \frac{1}{2}(x+x^{1/2})^{-1/2}\cdot(1+\frac{1}{2}x^{-1/2})$, or

$$h'(x) = \frac{1+\frac{1}{2}x^{-1/2}}{2\sqrt{x+\sqrt{x}}} = \frac{2\sqrt{x}+1}{4\sqrt{x}\sqrt{x+\sqrt{x}}} = \frac{2\sqrt{x}+1}{4\sqrt{x^2+x\sqrt{x}}}.$$

5. From $[\cos(y^2)+2x]' = (y^3)'$ we get

$$-\sin(y^2)\cdot 2yy' + 2 = 3y^2y' \quad\Rightarrow\quad 2yy'\sin(y^2) + 3y^2y' = 2 \quad\Rightarrow\quad y' = \frac{2}{2y\sin(y^2)+3y^2}.$$

6. From $(x^4)' = (2x^2+2y^2)'$ we get

$$4x^3 = 4x + 4yy' \quad\Rightarrow\quad y' = \frac{4x^3-4x}{4y} \quad\Rightarrow\quad y' = \frac{x^3-x}{y}.$$

Slope of tangent line is thus $y' = (2^3-2)/2 = 3$, so equation is $y-2 = 3(x-2)$, or $y = 3x-4$.

7. Area of rectangle at time t is $A(t) = (2+t)(4+t) = t^2 + 6t + 8$. Rate of change of the area at time t is $A'(t) = 2t + 6$. Thus at time $t = 20$ seconds the area is increasing at a rate of $A'(20) = 2(20) + 6 = 46\ \text{cm}^2/\text{s}$.

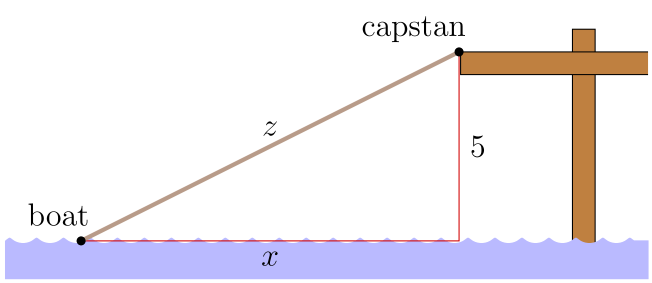
8. From the figure below we have $x^2 + 5^2 = z^2$, or $x = \sqrt{z^2 - 25}$. Here x and z are (implicitly) functions of time, and so we differentiate (implicitly) with respect to time to obtain

$$x'(t) = \frac{z(t)z'(t)}{\sqrt{z^2(t) - 25}}. \tag{1}$$

Now, we're given that $z'(t) = -3$ ft/s. Moreover, at the time t when $x(t) = 10$ ft we have $z(t) = 5\sqrt{5}$ ft by the Pythagorean Theorem. Putting these facts into (1) gives

$$x'(t) = \frac{(5\sqrt{5}\,)(-3)}{\sqrt{\left(5\sqrt{5}\,\right)^2 - 25}} = -\frac{3\sqrt{5}}{2}\ \text{ft/s}.$$

This tells us that the distance between the boat and the dock is *decreasing* at a rate of $\frac{3}{2}\sqrt{5}$ ft/s at the instant when the boat is 10 ft from the dock. Thus, when the boat is 10 ft from the dock it is traveling at a speed of $\frac{3}{2}\sqrt{5}$ ft/s.



9. We have $f'(x) = 3x^2 - 4x - 5$. Setting $f'(x) = 0$ gives the quadratic equation $3x^2 - 4x - 5 = 0$, which has solutions

$$x = \frac{2 \pm \sqrt{19}}{3} \approx 2.12,\ -0.79.$$

Neither of these critical points lies in $[4, 8]$, so we need only evaluate f at the endpoints of the interval: $f(4) = 18$ is the global minimum and $f(8) = 350$ the global maximum.

10a. $\text{Dom}(f) = \{x \in \mathbb{R} : x \neq -1\} = (-\infty, -1) \cup (-1, \infty)$.

10b. Since $f(0) = -1$, the y-intercept of f is $(0, -1)$. As for any x-intercepts, set $f(x) = 0$ and solve for x:

$$f(x) = 0 \quad \Rightarrow \quad \frac{x^3 - 1}{x^3 + 1} = 0 \quad \Rightarrow \quad x^3 - 1 = 0 \quad \Rightarrow \quad x = 1;$$

Thus f has x-intercept $(1, 0)$.

10c. We have

$$\lim_{x \to 1^-} \frac{x^3 - 1}{x^3 + 1} = +\infty \quad \text{and} \quad \lim_{x \to 1^+} \frac{x^3 - 1}{x^3 + 1} = -\infty,$$

so there is a vertical asymptote $x = -1$. Also

$$\lim_{x \to \pm\infty} \frac{x^3 - 1}{x^3 + 1} = 1,$$

so there is a horizontal asymptote $y = 1$.

10d. Differentiating f gives

$$f'(x) = \frac{(x^3 + 1)(3x^2) - (x^3 - 1)(3x^2)}{(x^3 + 1)^2} = \frac{6x^2}{(x^3 + 1)^2}.$$

It can be seen that $f' > 0$ on $(-\infty, -1) \cup (-1, 0) \cup (0, \infty)$, which shows that f is increasing everywhere on its domain except at 0.

The only critical point for f' that lies in $\text{Dom}(f)$ is $x = 0$, since $f'(0) = 0$. However, because $f' > 0$ for all x on $(-1, 0) \cup (0, \infty)$, no extremum lies at the corresponding point $(0, -1)$.

10e. Differentiating f' gives

$$f''(x) = \frac{(x^3 + 1)^2(12x) - (6x^2) \cdot 2(x^3 + 1)(3x^2)}{(x^3 + 1)^4} = \frac{12x(1 - 2x^3)}{(x^3 + 1)^3}.$$

Setting $f''(x) = 0$ implies that either $12x = 0$ or $1 - 2x^3 = 0$, from which we obtain solutions $x = 0, 2^{-1/3}$.

Now, $f''(-2) \approx 1.19 > 0$, and since $f''(x) \neq 0$ for all $x \in (-\infty, -1)$ and f'' is continuous on $(-\infty, -1)$, the Intermediate Value Theorem implies that $f'' > 0$ on $(-\infty, -1)$. Therefore f is concave up on $(-\infty, -1)$.

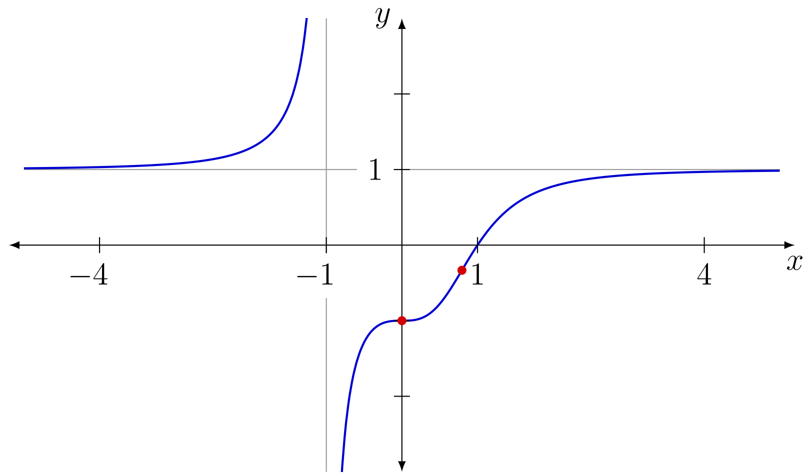
Next, $f''(-1/2) \approx -11.20 < 0$, and since $f''(x) \neq 0$ for all $x \in (-1, 0)$ and f'' is continuous on $(-1, 0)$, the IVT implies $f'' < 0$ on $(-1, 0)$. Therefore f is concave down on $(-1, 0)$.

A convenient value lying between 0 and $2^{-1/3}$ is $1/2$. We have $f''(1/2) \approx 3.16 > 0$, and since $f''(x) \neq 0$ for all $x \in (0, 2^{-1/3})$ and f'' is continuous on $(0, 2^{-1/3})$, the IVT implies $f'' > 0$ on $(0, 2^{-1/3})$. Therefore f is concave up on $(0, 2^{-1/3})$.

Finally, $f''(1) = -3/2 < 0$, and since $f''(x) \neq 0$ for all $x \in (2^{-1/3}, \infty)$ and f'' is continuous on $(2^{-1/3}, \infty)$, the IVT implies $f'' < 0$ on $(2^{-1/3}, \infty)$. Therefore f is concave down on $(2^{-1/3}, \infty)$.

The inflection points of f are $(0, -1)$ and $(2^{-1/3}, -1/3)$. Note that there is not an inflection point at $x = -1$ since f is undefined there!

10f. Inflection points are marked in red.



11. If x and y are the length and width of the garden, then $xy = 30$ and so $y = 30/x$. Meanwhile the combined area A of the garden and border is $(x+4)(y+2)$, or

$$A(x) = (x+4)\left(\frac{30}{x}+2\right) = 38 + 2x + \frac{120}{x}.$$

The goal is to find x so that $A(x)$ is minimized. We have

$$A'(x) = 2 - \frac{120}{x^2},$$

and so if we set $A'(x) = 0$ we obtain

$$2 - \frac{120}{x^2} = 0 \quad \Rightarrow \quad 2x^2 - 120 = 0 \quad \Rightarrow \quad x^2 = 60 \quad \Rightarrow \quad x = \sqrt{60} = 2\sqrt{15}$$

(obviously we must have $x > 0$). Thus the length of the garden should be $x = 2\sqrt{15}$ m, and the width should be $y = 30/x = 30/(2\sqrt{15}) = \sqrt{15}$ m, in order to minimize A. That is, the garden should have dimensions $2\sqrt{15}$ m $\times$ $\sqrt{15}$ m.

12. The base of the box is square, so let x be the length of the base edges (the length and width of the box), and let y be the height of box. The volume must be 16 ft^3, so $x^2y = 16$ and hence $y = 16/x^2$. Let k be the cost per square foot (in dollars) for the material in the sides of the box. The cost C of the box is a function of x as follows:

$$C(x) = \underbrace{\left(\frac{2k \text{ dollars}}{\text{ft}^2}\right)(x^2 \text{ ft}^2)}_{\text{cost of base}} + \underbrace{\left(\frac{0.5k \text{ dollars}}{\text{ft}^2}\right)(x^2 \text{ ft}^2)}_{\text{cost of top}} + \underbrace{4\left(\frac{k \text{ dollars}}{\text{ft}^2}\right)\left(x \cdot \frac{16}{x^2} \text{ ft}^2\right)}_{\text{cost of sides}},$$

and therefore

$$C(x) = \frac{5}{2}kx^2 + \frac{64k}{x}.$$

We wish to find x such that the cost is minimized. We have

$$C'(x) = 5kx - \frac{64k}{x^2},$$

and so if we set $C'(x) = 0$ we obtain

$$5kx - \frac{64k}{x^2} = 0 \quad \Rightarrow \quad 5kx^3 - 64k = 0 \quad \Rightarrow \quad 5x^3 = 64 \quad \Rightarrow \quad x = \frac{4}{\sqrt[3]{5}}.$$

Thus the cost is minimized if we set $x = 4/\sqrt[3]{5}$. That is, the box should have dimensions

$$\frac{4}{\sqrt[3]{5}} \text{ ft} \times \frac{4}{\sqrt[3]{5}} \text{ ft} \times \sqrt[3]{25} \text{ ft}.$$