Math 140 Exam #1 Key (Summer 2012)

1. It's more efficient to get the average velocity over the time interval $[0, h]$ first:

$$\Delta V_{[0,h]} = \frac{s(h) - s(0)}{h - 0} = \frac{(-4.9h^2 + 30h + 20) - 20}{h} = \frac{-4.9h^2 + 30h}{h} = -4.9h + 30.$$

Now, almost trivially, we get

$$\begin{aligned} \Delta V_{[0,2]} &= -4.9(2) + 30 = 20.2, \\ \Delta V_{[0,1]} &= -4.9(1) + 30 = 25.1, \\ \Delta V_{[0,0.5]} &= -4.9(0.5) + 30 = 27.55. \end{aligned}$$

2a. $\lim_{x\to 1^-} h(x) = 2$

2b. $\lim_{x\to 1^+} h(x) = 4$

2c. $\lim_{x\to 1} h(x) = \text{DNE}$

2d. $\lim_{x\to -2} h(x) = 1$

2e. $\lim_{x\to 3^-} h(x) = 1$

3a. $\sqrt[3]{t^2 - 10}$ is a composition of polynomial and radical functions, and 3 is in its domain. Therefore, by direct substitution,

$$\lim_{t\to 3} \sqrt[3]{t^2 - 10} = \sqrt[3]{3^2 - 10} = \sqrt[3]{-1} = -1.$$

3b. Here we factor to obtain a new function that behaves the same as the old one "near" 4:

$$\lim_{x\to 4} \frac{x^2 - 16}{4 - x} = \lim_{x\to 4} \frac{(x-4)(x+4)}{-(x-4)} = \lim_{x\to 4} (-x - 4) = -4 - 4 = -8.$$

3c. Multiply by conjugate of the numerator:

$$\begin{aligned} &\lim_{x\to 1} \frac{\sqrt{10x - 9} - 1}{x - 1} \cdot \frac{\sqrt{10x - 9} + 1}{\sqrt{10x - 9} + 1} = \lim_{x\to 1} \frac{(10x - 9) - 1}{(x - 1)(\sqrt{10x - 9} + 1)} \\ &= \lim_{x\to 1} \frac{10(x - 1)}{(x - 1)(\sqrt{10x - 9} + 1)} = \lim_{x\to 1} \frac{10}{\sqrt{10x - 9} + 1} = \frac{10}{\sqrt{10 - 9} + 1} = 5. \end{aligned}$$

4. For all $x \in (-\infty, 0) \cup (0, \infty)$ we have $-1 \le \sin(10/x) \le 1$, and thus

$$-x^2 \le x^2 \sin\left(\frac{10}{x}\right) \le x^2.$$

Now, since $\lim_{x\to 0} x^2 = \lim_{x\to 0}(-x^2) = 0$, it follows by the Squeeze Theorem that

$$\lim_{x\to 0} x^2 \sin\left(\frac{10}{x}\right) = 0$$

as well.

5. Factor numerator and denominator:

$$f(x) = \frac{(x-2)(x-7)}{(x-2)(x-3)}.$$

Now,

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x-7}{x-3} = \frac{2-7}{2-3} = 5,$$

shows that f does *not* have a vertical asymptote at $x=2$. On the other hand

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \frac{x-7}{x-3} = -\infty \quad \text{and} \quad \lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{x-7}{x-3} = \infty,$$

and so f does have a vertical asymptote at $x=3$. Since $\lim_{x\to 3^+} f(x) \neq \lim_{x\to 3^-} f(x)$, we can only say that $\lim_{x\to 3} f(x)$ does not exist.

6. Recall that in general $\sqrt{x^2} = |x|$. Now, when $x \to \infty$ we have $x > 0$, so then $\sqrt{x^2} = x$ and we obtain

$$\begin{aligned}\lim_{x\to\infty} \frac{\sqrt{x^2+2x+6}-3}{2x-1} &= \lim_{x\to\infty} \frac{x\sqrt{1+2/x+6/x^2}-3}{2x-1} \\ &= \lim_{x\to\infty} \frac{\sqrt{1+2/x+6/x^2}-3/x}{2-1/x} = \frac{\sqrt{1+0+0}-0}{2-0} = \frac{1}{2}.\end{aligned}$$

On the other hand $x \to -\infty$ implies $x < 0$, so then $\sqrt{x^2} = -x$ and we obtain

$$\begin{aligned}\lim_{x\to-\infty} \frac{\sqrt{x^2+2x+6}-3}{2x-1} &= \lim_{x\to-\infty} \frac{-x\sqrt{1+2/x+6/x^2}-3}{2x-1} \\ &= \lim_{x\to-\infty} \frac{-\sqrt{1+2/x+6/x^2}-3/x}{2-1/x} = \frac{-\sqrt{1+0+0}-0}{2-0} = -\frac{1}{2}.\end{aligned}$$

Hence the horizontal asymptotes of f are $y = \frac{1}{2}$ and $y = -\frac{1}{2}$.

7. f is continuous at 1 if and only if $\lim_{x\to 1} f(x) = f(1) = 1/1 = 1$; but

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (1-x^2) = 1 - 1^2 = 0 \neq f(1),$$

which is sufficient to show that $\lim_{x\to 1} f(x) \neq f(1)$, and therefore f is not continuous at 1.

8. Let $f(x) = x^3 - 5x^2 + 2x$, which is a continuous function on $(-\infty, \infty)$, and so certainly is continuous on $[-1, 5]$. Now, $f(-1) = (-1)^3 - 5(-1)^2 + 2(-1) = -8$ and $f(5) = 5^3 - 5(5^2) + 2(5) =$

10, and since -1 lies between $f(-1)$ and $f(5)$, the Intermediate Value Theorem implies that there exists some $c \in (-1, 5)$ for which $f(c) = -1$. That is, $c^3 - 5c^2 + 2c = -1$ for some $-1 < c < 5$, which shows that the equation $x^3 - 5x^2 + 2x = -1$ has a solution on the interval $(-1, 5)$.

9a. We find $f'(1)$:

$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{2(1+h)^3 - 2(1)^3}{h} = \lim_{h\to 0} \frac{2(1+3h+3h^2+h^3) - 2}{h} \\ &= \lim_{h\to 0} \frac{6h + 6h^2 + 2h^3}{h} = \lim_{h\to 0}(6 + 6h + 2h^2) = 6 + 6(0) + 2(0)^2 = 6. \end{aligned}$$

Thus the slope of the graph of f at $(1, 2)$ is 6.

9b. By the point-slope formula we have $y - 2 = 6(x - 1)$, or $y = 6x - 4$.

10a. $f'(t) = 30t^5 - 70t^9$

10b. By the Quotient Rule,

$$g'(z) = \frac{(5z-3)(2z+1)' - (2z+1)(5z-3)'}{(5z-3)^2} = \frac{(5z-3)(2) - (2z+1)(5)}{(5z-3)^2} = -\frac{11}{(5z-3)^2}.$$