Math 140 Exam #2 Key (Summer 2011)

1. $\displaystyle\lim_{x\to 0}\frac{\tan 9x}{\sin x}=\lim_{x\to 0}\frac{\sin 9x}{9x}\cdot\frac{9}{\cos 9x}\cdot\frac{x}{\sin x}=\lim_{x\to 0}\frac{\sin 9x}{9x}\cdot\lim_{x\to 0}\frac{9}{\cos 9x}\cdot\lim_{x\to 0}\frac{x}{\sin x}=1\cdot\frac{9}{1}\cdot 1=9$

2. $\displaystyle f'(x)=\frac{(1+\cos x)(x\sin x)'-(x\sin x)(1+\cos x)'}{(1+\cos x)^2}=\frac{(1+\cos x)(x\cos x+\sin x)-(x\sin x)(-\sin x)}{(1+\cos x)^2}$ $\displaystyle=\frac{\sin x\cos x+\sin x+x\cos x+x}{(1+\cos x)^2}=\frac{\sin x(1+\cos x)+x(1+\cos x)}{(1+\cos x)^2}=\frac{x+\sin x}{1+\cos x}$

3. $y'(x)=4\cos^2 x-4\sin^2 x$, so $y'(\pi/3)=4[\cos^2(\pi/3)-\sin^2(\pi/3)]=4[(1/2)^2-(\sqrt{3}/2)^2]=4(1/4-3/4)=-2$ is the slope of the tangent line, and the point of tangency is $(\pi/3, 4\sin\frac{\pi}{3}\cos\frac{\pi}{3})=(\pi/3,\sqrt{3})$. The equation of the tangent line is $y-\sqrt{3}=-2(x-\pi/3)$, or $y=-2x+2\pi/3+\sqrt{3}$.

4a. $f'(x)=28x^6\sec^2(4x^7)$

4b. $g'(x)=-20(3x^8+x)^{-5}\cdot(24x^7+1)$

4c. $h'(x)=\frac{1}{2}(x+\sqrt{x})^{-1/2}\cdot(x+\sqrt{x})'=\frac{1}{2}(x+\sqrt{x})^{-1/2}\cdot(1+\frac{1}{2}x^{-1/2})=\dfrac{1}{2\sqrt{x+\sqrt{x}}}\left(1+\dfrac{1}{2\sqrt{x}}\right)$

5. Differentiate both sides of equation with respect to x, treating y as a function of x and using the Chain Rule: $3(xy+1)^2\cdot(xy'+y)=1-2yy' \Rightarrow 3x(xy+1)^2y'+3y(xy+1)^2=1-2yy' \Rightarrow 3x(xy+1)^2y'+2yy'=1-3y(xy+1)^2 \Rightarrow y'=\dfrac{1-3y(xy+1)^2}{3x(xy+1)^2+2y}$, where y' is otherwise known as dy/dx.

6. Using implicit differentiation: $3x^2+3y^2y'=2y+2xy' \Rightarrow 3y^2y'-2xy'=2y-3x^2 \Rightarrow y'(x,y)=\dfrac{2y-3x^2}{3y^2-2x}$. Thus the tangent line to the curve has slope $y'(1,1)=\dfrac{2(1)-3(1)^2}{3(1)^2-2(1)}=-1$. Equation of line is $y-1=-(x-1) \Rightarrow y=-x+2$.

7. The equation $V(t)=\frac{4}{3}\pi r^3(t)$ reflects that the volume V of the sphere is ultimately a function of time t (since the radius r varies over time). We differentiate the function V to get $V'(t)=4\pi r^2(t)r'(t)$. Now, we're generously informed that $V'(t)=35$ cm^3/min for all t, and asked to find the value of $r'(t)$ at the time t when $r(t)=20$ cm. It should be noted for the record that we are not required to determine at *what* time this occurs, since the equation we've derived gives us $35=4\pi(20)^2r'(t) \Rightarrow r'(t)=\dfrac{35}{1600\pi}=\dfrac{7}{320\pi}\approx 0.0070$ cm/min.

8. Let $x(t)$ be the distance the base of the ladder is from the wall at time t, and let $y(t)$ be the distance the top of the ladder is from the ground. By the Pythagorean Theorem we have $y(t)=\sqrt{16^2-x^2(t)}$, and thus $y'(t)=-\dfrac{x(t)x'(t)}{\sqrt{256-x^2(t)}}$. We're given that $x'(t)=0.7$ ft/s for all t, and at the time t when $x(t)=10$ ft we have: $y'(t)=-\dfrac{(10)(0.7)}{\sqrt{256-10^2}}=-\dfrac{7}{2\sqrt{39}}$ ft/s ≈ -0.56 ft/s.

9. $f'(x) = \dfrac{6-4x}{x^3}$, so $f'(x) = 0$ gives $6 - 4x = 0 \Rightarrow x = 3/2$, and $f'(x) = \text{DNE}$ gives $x = 0$. The only critical point in $[1, 4]$ is therefore $3/2$. We evaluate: $f(1) = 1$, $f(4) = 13/16$, and $f(3/2) = 4/3$. The absolute maximum value of f on $[1, 4]$ is therefore $f(3/2) = 4/3$, and the absolute minimum value is $f(4) = 13/16$.

10a. From $g'(x) = 4x^2(x + 6)$ we find that $g'(x) = 0$ at $x = -6, 0$, and since $g' < 0$ on $(-\infty, -6)$, and $g' > 0$ on $(-6, 0) \cup (0, \infty)$, the First Derivative Test (FDT) implies that g is decreasing on $(-\infty, -6)$, increasing on $(-6, \infty)$, and has a local minimum at $g(-6) = -232$.

10b. From $g''(x) = 12x(x + 4)$ we find that $g''(x) = 0$ at $x = -4, 0$, and since $g'' > 0$ on $(-\infty, -4)$, $g'' < 0$ on $(-4, 0)$, and $g'' > 0$ on $(0, \infty)$, the Concavity Test implies that g is concave up on $(-\infty, -4)$, concave down on $(-4, 0)$, and concave up on $(0, \infty)$. There are inflection points at $g(-4) = -56$ and $g(0) = 200$.

11. We need to find $x, y > 0$ such that $xy = 50$ and $x + y$ is minimal. From $xy = 50$ we obtain $y = 50/x$. Now $x + y$ can be written $x + 50/x$. Letting $s(x) = x + 50/x$, we set about finding some $x > 0$ such that $s(x)$ is minimized. This looks like a job for Mr. FDT! Getting $s'(x) = 1 - 50/x^2$ and setting $s'(x) = 0$, we obtain critical points $x = 0$ and $x = \pm\sqrt{50} = \pm 5\sqrt{2}$. The requirement that x be positive disqualifies 0 and $-5\sqrt{2}$, so all we've got is $5\sqrt{2}$. Since $s' < 0$ on $(0, 5\sqrt{2})$ and $s' > 0$ on $(5\sqrt{2}, \infty)$, the FDT implies that $s(x)$ has a local minimum value at $x = 5\sqrt{2}$. The two Magic Numbers are $x = 5\sqrt{2}$ and $y = 50/(5\sqrt{2}) = 50/\sqrt{50} = \sqrt{50} = 5\sqrt{2}$, which are the same number.

12. This one is #4.4.13 in the book and has been done in class. The answer is $4/\sqrt[3]{5}$ ft $\times$ $4/\sqrt[3]{5}$ ft $\times$ $\sqrt[3]{25}$ ft.