Math 140 Exam #1 Key (Summer 2011)

1. $\Delta V_{[0,3]} = 15.3$, $\Delta V_{[0,2]} = 20.2$, $\Delta V_{[0,1]} = 25.1$, $\Delta V_{[0,h]} = \dfrac{-4.9h^2 + 30h}{h} = -4.9h + 30$

2. (a) $\lim\limits_{x\to -1} g(x) = 2$; (b) $\lim\limits_{x\to 1^-} g(x) = 4$; (c) $\lim\limits_{x\to 1^+} g(x) = 5$; (d) $\lim\limits_{x\to 1} g(x)$ D.N.E. since $\lim_{x\to 1^-} g(x) \neq \lim_{x\to 1^+} g(x)$; (e) $\lim\limits_{x\to 5^-} g(x) = 5$.

3a. Since the domain of the function is $(2,3]$ and the limit approaches 3 from the left, we can employ direct substitution: $\lim\limits_{x\to 3^-} \sqrt{\dfrac{x-3}{2-x}} = \sqrt{\dfrac{3-3}{2-3}} = 0$.

3b. $\lim\limits_{x\to 3} \dfrac{(x-3)(x+1)}{x-3} = \lim\limits_{x\to 3}(x+1) = 3+1 = 4$.

3c. $\lim\limits_{x\to 25} \dfrac{\sqrt{x}-5}{x-25} = \lim\limits_{x\to 25} \dfrac{\sqrt{x}-5}{(\sqrt{x}-5)(\sqrt{x}+5)} = \lim\limits_{x\to 25} \dfrac{1}{\sqrt{x}+5} = \dfrac{1}{\sqrt{25}+5} = \dfrac{1}{10}$.

4. $\lim\limits_{x\to 7^+} \dfrac{9}{x-7} = +\infty$; $\lim\limits_{x\to 7^-} \dfrac{9}{x-7} = -\infty$; $\lim\limits_{x\to 7} \dfrac{9}{x-7} =$ D.N.E.

5. For all $x \neq 0$ we have $-1 \le \cos\frac{2}{x} \le 1$, which we can multiply through by x^4 to get $-x^4 \le x^4\cos\frac{2}{x} \le x^4$. Then, defining $f(x) = -x^4$, $g(x) = x^4\cos\frac{2}{x}$, and $h(x) = x^4$, and noting that we have $f(x) \le g(x) \le h(x)$ and $\lim_{x\to 0} f(x) = \lim_{x\to 0} h(x) = 0$, it follows from the Squeeze Theorem that $\lim_{x\to 0} x^4\cos\frac{2}{x} = \lim_{x\to 0} g(x) = 0$.

6. Recall that in general $\sqrt{x^2} = |x|$. Now, when $x \to \infty$ we have $x > 0$, so then $\sqrt{x^2} = x$ and we obtain $\lim\limits_{x\to\infty} \dfrac{\sqrt{x^2+2x+6}-3}{x-1} = \lim\limits_{x\to\infty} \dfrac{x\sqrt{1+2/x+6/x^2}-3}{x-1} = \lim\limits_{x\to\infty} \dfrac{\sqrt{1+2/x+6/x^2}-3/x}{1-1/x} = \dfrac{\sqrt{1+0+0}-0}{1-0} = 1$. On the other hand $x \to -\infty$ implies $x < 0$, so then $\sqrt{x^2} = -x$ and we obtain $\lim\limits_{x\to-\infty} \dfrac{\sqrt{x^2+2x+6}-3}{x-1} = \lim\limits_{x\to-\infty} \dfrac{-x\sqrt{1+2/x+6/x^2}-3}{x-1} = \lim\limits_{x\to-\infty} \dfrac{-\sqrt{1+2/x+6/x^2}-3/x}{1-1/x} = \dfrac{-\sqrt{1+0+0}-0}{1-0} = -1$. Hence the horizontal asymptotes of f are $y = 1$ and $y = -1$.

A vertical asymptote *might* lie at $x = 1$; however, since $\lim\limits_{x\to 1}\left(\dfrac{\sqrt{x^2+2x+6}-3}{x-1} \cdot \dfrac{\sqrt{x^2+2x+6}+3}{\sqrt{x^2+2x+6}+3}\right) = \lim\limits_{x\to 1} \dfrac{(x+3)(x-1)}{(x-1)(\sqrt{x^2+2x+3}+3)} = \lim\limits_{x\to 1} \dfrac{x+3}{\sqrt{x^2+2x+3}+3} = \dfrac{1+3}{\sqrt{1^2+2\cdot 1+3}+3} = \dfrac{4}{\sqrt{6}+3} \neq \pm\infty$, this is not the case.

7. The function f is not continuous at 1 since $\lim\limits_{x\to 1} f(x) \neq f(1) = 1$, as can be seen by evaluating one-sided limits: $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} 1/x = 1$ and $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-}(1-x^2) = 0$.

8. Let $f(x) = 2x^3 + x - 2$, so f is a polynomial function and thus continuous everywhere, which includes the interval $[-1,1]$. Now, since $f(-1) = -5$, $f(1) = 1$, and 0 lies between -5 and 1, the Intermediate Value Theorem implies that there exists some $c \in (-1,1)$ such that $f(c) = 0$. Therefore $2c^3 + c - 2 = 0$ and the equation has at least one solution in $(-1,1)$.

9a. Either version of the definition will do: $\lim_{x\to-1} \dfrac{f(x)-f(-1)}{x-(-1)} = \lim_{x\to-1} \dfrac{\frac{1}{3-2x} - \frac{1}{5}}{x+1} = \lim_{x\to-1} \dfrac{5-(3-2x)}{(x+1)\cdot 5(3-2x)} = \lim_{x\to-1} \dfrac{2(x+1)}{(x+1)(15-10x)} = \lim_{x\to-1} \dfrac{2}{15-10x} = \dfrac{2}{25}$.

9b. Slope of line is $2/25$ and it contains the point $(-1, 1/5)$, so point-slope formula yields $y - \frac{1}{5} = \frac{2}{25}(x+1)$, or $y = \frac{2}{25}x + \frac{7}{25}$.

10a. $g'(w) = -27w^{-10}$

10b. $f'(t) = 48t^7 - 3t^2$

10c. $h'(x) = \dfrac{(x^2-1)(4x^3) - (x^4+1)(2x)}{(x^2-1)^2} = \dfrac{2x^5 - 4x^3 - 2x}{(x^2-1)^2}$