MATH 140 EXAM #3 KEY (SUMMER 2010)

1a. $s'(t) = 12t^3 + 12t^2 - 12t = 12t(t^2 + t - 1)$, so $s'(t) = 0$ implies that $t = 0, \dfrac{-1 \pm \sqrt{5}}{2}$ are the critical numbers.

1b. $f'(x) = \dfrac{4}{5}x^{-1/5}(x-4)^2 + x^{4/5} \cdot 2(x-4) = 2(x-4)\left[x^{4/5} + \dfrac{2}{5}x^{-1/5}(x-4)\right] = \dfrac{2}{5}(x-4)(7x-8)x^{-1/5}$, so the critical numbers are $x = 0,\ 4,\ 8/7$.

2a. $f'(x) = 4x^3 - 4x = 4x(x^2-1) = 4x(x-1)(x+1)$, so the critical numbers are $0,\ 1,\ -1$. We evaluate: $f(-2) = 11$, $f(-1) = 2$, $f(0) = 3$, $f(1) = 2$, $f(3) = 66$. Absolute maximum is $f(3) = 66$, and absolute minimum is $f(-1) = f(1) = 2$.

2b. $f'(x) = \cos x - \sin x$. Now, $f'(x) = 0 \Rightarrow \cos x = \sin x \Rightarrow \tan x = 1 \Rightarrow x = \dfrac{\pi}{4}$, which is the only solution that fits in the interval $\left[0, \dfrac{\pi}{3}\right]$. Now, $f(0) = 1$, $f\left(\dfrac{\pi}{4}\right) = \dfrac{2}{\sqrt{2}} = \sqrt{2}$, $f\left(\dfrac{\pi}{3}\right) = \sin\dfrac{\pi}{3} + \cos\dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2} + \dfrac{1}{2} = \dfrac{\sqrt{3}+1}{2}$. Absolute maximum: $f\left(\dfrac{\pi}{4}\right) = \sqrt{2}$, absolute minimum: $f(0) = 1$.

3. f is continuous on $(-\infty,-2) \cup (-2,\infty)$, which contains the interval $[1,4]$. Also f is differentiable on $(-\infty,-2) \cup (-2,\infty)$, which contains the interval $(1,4)$. By the Mean Value Theorem there exists some $c \in (1,4)$ such that $f'(c) = \dfrac{f(4)-f(1)}{4-1} = \dfrac{1}{3}\left(\dfrac{2}{3} - \dfrac{1}{3}\right) = \dfrac{1}{9}$. Now, $f'(x) = \dfrac{2}{(x+2)^2}$, so $f'(c) = \dfrac{1}{9}$ implies that $\dfrac{2}{(c+2)^2} = \dfrac{1}{9}$, or $c = -2 \pm 3\sqrt{2}$. Now, notice that $-2 + 3\sqrt{2}$ lies in $(1,4)$.

4a. $h'(x) = 15x^2(x^2-1)$, which should make clear that $h'(x) > 0$ on $(-\infty,-1) \cup (1,\infty)$ and $h'(x) < 0$ on $(-1,0) \cup (0,1)$; thus, $h(x)$ is increasing on $(-\infty,-1) \cup (1,\infty)$ and decreasing on $(-1,0) \cup (0,1)$.

4b. Local maximum is $h(-1) = 5$, and local minimum is $h(1) = 1$.

4c. $h''(x) = 60x^3 - 30x = 30x(\sqrt{2}x - 1)(\sqrt{2}x + 1)$, so $h''(x) > 0$ on $\left(-\dfrac{1}{\sqrt{2}}, 0\right) \cup \left(\dfrac{1}{\sqrt{2}}, \infty\right)$ and $h''(x) < 0$ on $\left(-\infty, -\dfrac{1}{\sqrt{2}}\right) \cup \left(0, \dfrac{1}{\sqrt{2}}\right)$. So $h(x)$ is concave up on $\left(-\dfrac{1}{\sqrt{2}}, 0\right) \cup \left(\dfrac{1}{\sqrt{2}}, \infty\right)$ and concave down on $\left(-\infty, -\dfrac{1}{\sqrt{2}}\right) \cup \left(0, \dfrac{1}{\sqrt{2}}\right)$. Inflection points are at $x = -\dfrac{1}{\sqrt{2}},\ 0,\ \dfrac{1}{\sqrt{2}}$.

5a. $= \displaystyle\lim_{y\to\infty} \frac{-3y^2+2}{5y^2+4y} = -\frac{3}{5}.$

5b. $= \displaystyle\lim_{x\to-\infty} \frac{\sqrt{x^6(9-x^{-5})}}{x^3+1} = \lim_{x\to-\infty} \frac{|x|^3\sqrt{9-1/x^5}}{x^3+1} = \lim_{x\to-\infty} \frac{-x^3\sqrt{9-1/x^5}}{x^3+1} = \lim_{x\to-\infty} \frac{-\sqrt{9-1/x^5}}{1+1/x^3} = \frac{-\sqrt{9}}{1} = -3.$

5c. $= \displaystyle\lim_{x\to\infty} \left(\frac{\sqrt{x^4+6x^2} - x^2}{1} \cdot \frac{\sqrt{x^4+6x^2}+x^2}{\sqrt{x^4+6x^2}+x^2}\right) = \lim_{x\to\infty} \frac{6x^2}{\sqrt{x^4+6x^2}+x^2} = \lim_{x\to\infty} \frac{6x^2}{x^2\sqrt{1+6/x^2}+x^2}$

$= \displaystyle\lim_{x\to\infty} \frac{6}{\sqrt{1+6/x^2}+1} = \frac{6}{\sqrt{1+0}+1} = 3.$

6. A point on the line has coordinates $(x, 2x-9)$. The distance between $(x, 2x-9)$ and $(5,-2)$ is given by $D(x) = \sqrt{(x-5)^2 + ((2x-9)-(-2))^2} = \sqrt{5x^2 - 38x + 74}$. Now, $D'(x) = \frac{1}{2}(5x^2 - 38x + 74)^{-1/2} \cdot (10x - 38)$, and it's seen that $D'(x) = 0$ only when $x = \frac{38}{10}$. That is, the point $(\frac{19}{5}, 2(\frac{19}{5}) - 9) = (\frac{19}{5}, -\frac{7}{5})$ is the closest to $(5,-2)$.

7. $V = \pi r^2 h$, where V is regarded as a *constant*. We write $h = \dfrac{V}{\pi r^2}$ and thereby eliminate the variable h. The surface area S of the can is given by $S = \pi r^2 + 2\pi r h$, which then gives the function $S(r) = \pi r^2 + \dfrac{2V}{r}$. We want to find the minimum value for this function, so we obtain its derivative: $\dfrac{dS}{dr} = 2\pi r - \dfrac{2V}{r^2}$. The derivative does not exist when $r = 0$, but we can't have a can with zero radius so forget this critical number. Setting $\dfrac{dS}{dr} = 0$ and solving yields: $2\pi r - \dfrac{2V}{r^2} = 0$, or $r = \sqrt[3]{\dfrac{V}{\pi}}$. This is what we want. Dimensions of the can: radius of $\sqrt[3]{\dfrac{V}{\pi}}$, height of $\sqrt[3]{\dfrac{V}{\pi}}$ (put $r = \sqrt[3]{\dfrac{V}{\pi}}$ into $h = \dfrac{V}{\pi r^2}$).

8. $\displaystyle\int \left(6x^{1/2} - x^{1/6}\right) dx = 6 \cdot \frac{2}{3}x^{3/2} - \frac{6}{7}x^{7/6} + C = 4x^{3/2} - \frac{6}{7}x^{7/6} + C.$

9. $f'(x) = \displaystyle\int f''(x)dx = 5x^4 + 4x^3 + 4x + C \Rightarrow f(x) = \int f'(x)dx = x^5 + x^4 + 2x^2 + Cx + D$. Now, $f(0) = 8$ implies that $D = 8$, giving us $f(x) = x^5 + x^4 + 2x^2 + Cx + 8$. Next, $f(1) = 5$ gives $1 + 1 + 2 + C + 8 = 5$, or $C = -7$. Therefore $f(x) = x^5 + x^4 + 2x^2 - 7x + 8$.

10. Let $f(x) = 4x^5 + x^3 + 2x + 1$, which is a function that is continuous and differentiable on $(-\infty, \infty)$. Now, $f(-1) = -6 < 0$ and $f(0) = 1 > 0$, so by the Intermediate Value Theorem there exists some $c \in (-1, 0)$ such that $f(c) = 0$ (i.e. $4c^5 + c^3 + 2c + 1 = 0$). This demonstrates that the equation has *at least one* real root.

Now, suppose f has two real roots c_1 and c_2. Then $f(c_1) = 0$ and $f(c_2) = 0$. By the Mean Value Theorem (or just apply its corollary, Rolle's Theorem) there exists some number b between c_1 and c_2 such that $f'(b) = 0$. Thus $20b^4 + 3b^2 + 2 = 0$. But we can see that we must have $f'(x) \geq 2$ for all $x \in (-\infty, \infty)$ since $20x^4 \geq 0$ and $3x^2 \geq 0$ always hold for $f'(x) = 20x^4 + 3x^2 + 2$. So $f'(b) = 0$ for some real number b is a contradiction. Hence f cannot have two real roots.

Therefore f must have *exactly* one real root. ∎