MATH 140 EXAM #2 KEY (SUMMER 2010)

1. $g'(x) = \lim_{h\to 0} \dfrac{\sqrt{1+2(x+h)} - \sqrt{1+2x}}{h} = \lim_{h\to 0} \dfrac{\sqrt{1+2(x+h)} - \sqrt{1+2x}}{h} \cdot \dfrac{\sqrt{1+2(x+h)} + \sqrt{1+2x}}{\sqrt{1+2(x+h)} + \sqrt{1+2x}}$

$= \lim_{h\to 0} \dfrac{1+2(x+h)-(1+2x)}{h[\sqrt{1+2(x+h)} + \sqrt{1+2x}\,]} = \lim_{h\to 0} \dfrac{2h}{h[\sqrt{1+2(x+h)} + \sqrt{1+2x}\,]} = \lim_{h\to 0} \dfrac{2}{\sqrt{1+2(x+h)} + \sqrt{1+2x}}$

$= \dfrac{2}{\sqrt{1+2x} + \sqrt{1+2x}} = \dfrac{1}{\sqrt{1+2x}}.$ So Dom $g = [\frac{1}{2}, \infty)$, and Dom $g' = (\frac{1}{2}, \infty)$.

2a. We have $f(x) = x^{1/2}(x^2-1) = x^{5/2} - x^{1/2}$, so $f'(x) = \dfrac{5}{2}x^{3/2} - \dfrac{1}{2}x^{-1/2}$.

2b. $g'(x) = \dfrac{(1+\sqrt{r})(2r) - r^2/(2\sqrt{r})}{(1+\sqrt{r})^2} = \dfrac{2r + 2r^{3/2} - \frac{1}{2}r^{3/2}}{(1+\sqrt{r})^2} = \dfrac{4r + 3r^{3/2}}{2(1+r^{1/2})^2}.$

2c. $h'(\theta) = \theta(-\csc\theta\cot\theta) + \csc\theta - (-\csc^2\theta) = -\theta\csc\theta\cot\theta + \csc\theta + \csc^2\theta.$

2d. $y' = \dfrac{1}{4}(1+2x+x^3)^{-3/4} \cdot (2+3x^2) = \dfrac{2+3x^2}{4(1+2x+x^3)^{3/4}}.$

2e. $y' = \cos(\tan 2x) \cdot \sec^2(2x) \cdot 2 = 2\sec^2(2x)\cos(\tan 2x).$

3. If we let $f(x) = x + \cos x$, then $f'(x) = 1 - \sin x$ and the slope of the tangent line at $(0,1)$ is $f'(0) = 1 - \sin(0) = 1$. Using the point-slope formula yields the equation $y - 1 = 1 \cdot (x - 0)$, or $y = x + 1$.

4. Implicit differentiation gives $y'\cos x - y\sin x = \cos(xy)\cdot(xy' + y)$, so $y'\cos x - xy'\cos(xy) = y\sin x + y\cos(xy)$ and we obtain $\dfrac{dy}{dx} = \dfrac{y\sin x + y\cos(xy)}{\cos x - x\cos(xy)}$.

5. Implicit differentiation of the equation yields $4(x^2+y^2)\cdot(2x+2yy') = 25(2x-2yy')$, whence $8yy'(x^2+y^2)+50yy' = 50x - 8x(x^2+y^2)$ and we get $y' = \dfrac{50x - 8x(x^2+y^2)}{8y(x^2+y^2)+50y}$. Now, the slope of the tangent line to the curve at $(3,1)$ is $m = \dfrac{50(3) - 8(3)(3^2+1^2)}{8(1)(3^2+1^2)+50(1)} = -\dfrac{9}{13}$. The point-slope formula gives $y - 1 = -\dfrac{9}{13}(x-3)$ for the equation of the tangent line, or $y = -\dfrac{9}{13}x + \dfrac{40}{13}$.

6a. Velocity v at time t is given by $v(t) = s'(t) = 5 + 6t$, so at 2 seconds velocity is: $v(2) = 5 + 6(2) = 17$ m/s.

6b. Find t so that $v(t) = 35$: $5 + 6t = 35 \Rightarrow t = 5$ s.

7. Story problem! At time t the westbound car has gone a distance of $42t$ miles while the southbound car has gone a distance of $70t$ miles. These distances are the lengths of the two legs of a right triangle, and the distance between the cars, $d(t)$ equals the length of the hypotenuse. By the Pythagorean Theorem we have $d(t) = \sqrt{(42t)^2 + (70t)^2}$, so the rate of change of the distance between the cars at time t is given by $d'(t) = \frac{1}{2}[(42t)^2 + (70t)^2]^{-1/2} \cdot [2\cdot 42^2 t + 2\cdot 70^2 t] = \dfrac{42^2 t + 70^2 t}{\sqrt{(42t)^2 + (70t)^2}}$. So at 3 hours the cars are moving apart at a rate of $d'(3) = \dfrac{42^2(3) + 70^2(3)}{\sqrt{(42\cdot 3)^2 + (70\cdot 3)^2}} = \dfrac{19{,}992}{\sqrt{59{,}976}} =$ 81.6 mph.

8. The diameter of the pile's base equals the height, which is to say the radius r equals $\frac{1}{2}h$ so that $V = \dfrac{1}{3}\pi\left(\dfrac{h}{2}\right)^2 h = \dfrac{\pi}{12}h^3$. Differentiating with respect to t gives $\dfrac{dV}{dt} = \dfrac{\pi}{4}h^2\dfrac{dh}{dt} \Rightarrow \dfrac{dh}{dt} = \dfrac{4}{\pi h^2}\dfrac{dV}{dt}$. But we're given that $\dfrac{dV}{dt} = 30$ ft^3/min, so we have $\dfrac{dh}{dt} = \dfrac{120}{\pi h^2}$. Finally we can find the rate at which the height of the pile is changing over time when $h = 12$ ft: $\left.\dfrac{dh}{dt}\right|_{h=12} = \dfrac{120}{\pi 12^2} = \dfrac{5}{6\pi} = 0.265$ ft/min.

9. The linearization of f at 0 is simply the tangent line to the curve given by $f(x) = 1/\sqrt{2+x}$ at the point $(0, f(0)) = (0, 1/\sqrt{2})$. The slope of the line is figured from $f'(x) = -\dfrac{1}{2}(2+x)^{-3/2}$ as $f'(0) = -\dfrac{1}{2} \cdot 2^{-3/2} = -2^{-5/2}$. The point-slope formula gives an equation for the tangent line, $y - 2^{-1/2} = -2^{-5/2}(x-0)$, which simplifies as $y = -\dfrac{1}{4\sqrt{2}}x + \dfrac{1}{\sqrt{2}}$. Thus $L(x) = -\dfrac{1}{4\sqrt{2}}x + \dfrac{1}{\sqrt{2}}$, or approximately $L(x) = -0.1768x + 0.7071$.

10. We have the function $f(x) = \sin x$. The tangent line to the curve given by $f(x) = \sin x$ at the point $(0, f(0)) = (0,0)$ will provide a reasonable linearization of the sine function for the purpose of estimating $\sin 1^\circ$. The slope of the tangent line is $f'(0) = \cos(0) = 1$, which gives us an equation for the tangent line: $y = x$. That is, $L(x) = x$ is our linearization, and close to 0 we can expect the value of $L(x)$ to be fairly close to $\sin x$. The trick, however, is that we must work in radians: $\sin 1^\circ = \sin(\pi/180) \approx L(\pi/180) = \pi/180$. This is a decent approximation, since $\pi/180 = 0.0174532925...$ while $\sin 1^\circ = 0.0174524064...$