MATH 140 EXAM #1 KEY (SUMMER 2010)

1a. $\displaystyle\lim_{x\to-2^+}\frac{x-1}{x^2(x+2)}=-\infty$

1b. $\displaystyle\lim_{x\to-1}\frac{x^2-4x}{x^2-3x-4}=\lim_{x\to-1}\frac{x(x-4)}{(x+1)(x-4)}=\lim_{x\to-1}\frac{x}{x+1}=$ DNE, since the one-sided limits are not equal $\left(\displaystyle\lim_{x\to-1^+}\frac{x}{x+1}=-\infty\right.$ while $\left.\displaystyle\lim_{x\to-1^-}\frac{x}{x+1}=+\infty\right)$.

1c. $\displaystyle\lim_{x\to7}\frac{\sqrt{x+2}-3}{x-7}=\lim_{x\to7}\frac{\sqrt{x+2}-3}{x-7}\cdot\frac{\sqrt{x+2}+3}{\sqrt{x+2}+3}=\lim_{x\to7}\frac{x-7}{(x-7)(\sqrt{x+2}+3)}=\lim_{x\to7}\frac{1}{\sqrt{x+2}+3}=\frac{1}{6}$.

1d. $\displaystyle\lim_{x\to3}(2x+|x-3|)=2(3)+0=6$, where it's necessary to work with one-sided limits to resolve the absolute value.

2. Let $f(x)=2x$ and $h(x)=x^4-x^2+2$. Note that $\lim_{x\to1}f(x)=\lim_{x\to1}h(x)=2$. Since $f(x)\le g(x)\le h(x)$ for all x (and in particular in a neighborhood of 1), the Squeeze Theorem implies that $\lim_{x\to1}g(x)=2$ also.

3. For all $x\neq0$ we have $-1\le\cos\frac{2}{x}\le1$, which we can multiply through by x^4 to get $-x^4\le x^4\cos\frac{2}{x}\le x^4$. We can define $f(x)=-x^4$, $g(x)=x^4\cos\frac{2}{x}$, and $h(x)=x^4$ if desired, and note that we have the following: $f(x)\le g(x)\le h(x)$ and $\lim_{x\to0}f(x)=\lim_{x\to0}h(x)=0$. By the Squeeze Theorem, then $\lim_{x\to0}x^4\cos\frac{2}{x}=\lim_{x\to0}g(x)=0$.

4a. Let $\epsilon>0$. (*Writing in the margin: we need to find some* $\delta>0$ *such that when we have* $0<|x-3|<\delta$*, it follows that* $|(2x+7)-13|<\epsilon$*. Manipulate the second inequality to bring* $|x-3|$ *out in the open:* $|(2x+7)-13|<\epsilon\Rightarrow|2(x-3)|<\epsilon\Rightarrow|x-3|<\epsilon/2$*. This suggests that we choose* $\delta=\epsilon/2$*.*) Back to the proof: choose $\delta=\epsilon/2$. Now, suppose that $0<|x-3|<\delta$. Then $|x-3|<\epsilon/2$, and so multiplying by 2 gives $2|x-3|<\epsilon\Rightarrow|2x-6|<\epsilon\Rightarrow|(2x+7)-13|<\epsilon$. This completes the proof.

4b. Let $\epsilon>0$ (*Some beer coaster scribblings: we need some* $\delta>0$ *such that whenever* $0<|x-2|<\delta$*, it follows that* $|(5-7x)-(-9)|<\epsilon$*. Manipulate the second inequality to bring out* $|x-2|$*:* $|(5-7x)-(-9)|=|5-7x+9|=|-7x+14|=|-7(x-2)|=|-7|\cdot|x-2|=7|x-2|$*, so* $|(5-7x)-(-9)|<\epsilon$ *becomes* $7|x-2|<\epsilon$*, which yields* $|x-2|<\epsilon/7$*. The Gods of Mathematics are telling us to choose* δ *to be* $\epsilon/7$*.*) Back to the proof: choose $\delta=\epsilon/7$. Suppose that $0<|x-2|<\delta$. Then $|x-2|<\epsilon/7$, and so multiplying by 7 gives $7|x-2|<\epsilon\Rightarrow|7x-14|<\epsilon\Rightarrow|14-7x|<\epsilon\Rightarrow|(5-7x)-(-9)|<\epsilon$. This is what we needed to show.

5. The function f is not continuous at 1 since $\lim_{x\to1}f(x)\neq f(1)=1$, as can be seen by evaluating one-sided limits: $\lim_{x\to1^+}f(x)=\lim_{x\to1^+}1/x=1$ and $\lim_{x\to1^-}f(x)=\lim_{x\to1^-}(1-x^2)=0$.

6. The function φ is discontinuous only at 1 and 3: $\lim_{x\to1}\varphi(x)\neq\varphi(1)=2$ since $\lim_{x\to1^+}\varphi(x)=\lim_{x\to1^+}1/x=1$, and $\lim_{x\to3}\varphi(x)\neq\varphi(3)=0$ since $\lim_{x\to3^-}\varphi(x)=\lim_{x\to3^-}1/x=1/3$. We see φ is continuous from the left at 1 and continuous from the right at 3.

7. Define $f(x)=\sqrt{x-5}-\frac{1}{x+3}$. The function f is continuous on its domain $[5,\infty)$, so certainly it is continuous on the interval $[5,6]$. Now, $f(5)=0-1/8=-1/8<0$ and $f(6)=1-1/9=8/9>0$, so 0 lies between $f(5)$ and $f(6)$. By ye olde Intermediate Value Theorem, then, there exists some $c\in(5,6)$ such that $f(c)=0$. Now, $f(c)=0\Rightarrow\sqrt{c-5}-\frac{1}{c+3}=0\Rightarrow\sqrt{c-5}=\frac{1}{c+3}$, which shows that c is a real root of the equation $\sqrt{x-5}=\frac{1}{x+3}$. Therefore the equation has at least one real root.

8. Let $f(x)=\dfrac{x-1}{x-2}$. The tangent line will contain the point $(3,2)$ and have slope $f'(3)=\displaystyle\lim_{h\to0}\frac{f(3+h)-f(3)}{h}=\lim_{h\to0}\frac{[(3+h)-1]/[(3+h)-2]-2}{h}=\lim_{h\to0}\frac{(2+h)/(1+h)-2}{h}=\lim_{h\to0}\frac{(2+h)-2(1+h)}{h(1+h)}=\lim_{h\to0}\frac{2+h-2-2h}{h(1+h)}=\lim_{h\to0}\frac{-h}{h(1+h)}=\lim_{h\to0}\frac{-1}{1+h}=-1$. The point-slope formula then gives the equation of the line: $y-2=-(x-3)$, or $y=-x+5$.

9. Let $s(t)=40t-16t^2$. The velocity of the ball at time $t=2$ equals $s'(2)=\lim_{h\to0}\dfrac{s(2+h)-s(2)}{h}=\displaystyle\lim_{h\to0}\frac{40(2+h)-16(2+h)^2-16}{h}=\lim_{h\to0}\frac{80+40h-64-64h-16h^2-16}{h}=\lim_{h\to0}\frac{-24h-16h^2}{h}=\lim_{h\to0}(-24-16h)=-24$ ft/s.