Math 140 Exam #3 Key (Spring 2024)

1 $f'(x) = 12x^5 - 60x^3 + 48x = 12x(x-2)(x+2)(x-1)(x+1)$, so $f'(x) = 0$ on $(-2,2)$ if $x = 0, -1, 1$. Now, $f(0) = 0$, $f(\pm 1) = 11$, and $f(\pm 2) = -16$, so f has an absolute maximum value of 11 at $x = \pm 1$, and an absolute minimum value of -16 at $x = \pm 2$.

2a $D_f = \{x : x \neq -\frac{1}{2}\}$; y-intercept is 12; no x-intercepts.

2b Vertical asymoptote is $x = -\frac{1}{2}$, slant asymptote is $y = \frac{1}{2}x - \frac{1}{4}$.

2c Obtaining

$$f'(x) = \frac{(2x+1)(2x) - (x^2+12)(2)}{(2x+1)^2} = \frac{2(x-3)(x+4)}{(2x+1)^2},$$

we find the critical points are $x = -4, 3$.

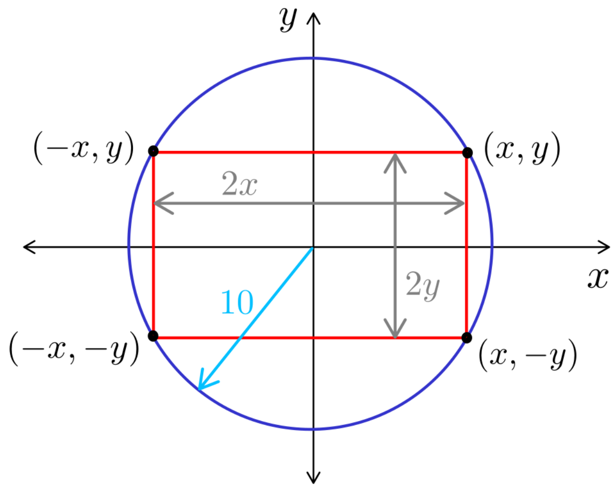
2d $f'(x) > 0$ implies $x \in (-\infty, -4) \cup (3, \infty)$, while $f'(x) < 0$ implies $x \in (-4, 3)$ with $x \neq -\frac{1}{2}$. So f is increasing on $(-\infty, -4)$ and $(3, \infty)$, decreasing on $(-4, -\frac{1}{2})$ and $(-\frac{1}{2}, 3)$. There is a local maximum point $(-4, f(-4)) = (-4, -4)$, and a local maximum point $(3, f(3)) = (3, 3)$.

2e Obtainin $f''(x) = 98/(2x+1)^3$, we have $f''(x) > 0$ for $x \in (-\frac{1}{2}, \infty)$, and $f''(x) < 0$ for $x \in (-\infty, -\frac{1}{2})$. Graph of f is concave up on $(-\frac{1}{2}, \infty)$, concave down on $(-\infty, -\frac{1}{2})$.

3 Center the rectangle and circle of radius 10 at the origin, as in the figure below. With (x, y) being the point in the 1st quadrant where the rectangle and circle intersect, the rectangle has length $2x$ and width $2y$. Since $x^2 + y^2 = 100$, so that $y = \sqrt{100 - x^2}$, the area $4xy$ of the rectangle becomes $A(x) = 4x\sqrt{100 - x^2}$ for $0 < x < 10$. Now,

$$A'(x) = 4\sqrt{100 - x^2} - \frac{4x^2}{\sqrt{100 - x^2}},$$

so that $A'(x) = 0$ implies $4(100 - x^2) - 4x^2 = 0$, and hence $x = 5\sqrt{2}$ is the only critical point in the domain $(0, 10)$ for A. We have $y = 5\sqrt{2}$ when $x = 5\sqrt{2}$, and so the rectangle with maximal area is a $5\sqrt{2} \times 5\sqrt{2}$ square.



4 Let h be the heigth and r the radius of the can. Volume V is the cross-sectional area times the height: $V = \pi r^2 h$. But also $V = 500$, so that $h = \frac{500}{\pi} r^{-2}$. We want to minimize the surface area of the can. Removing the bottom and slicing longitudinally, the side of the can may be uncurled to form a rectangle $2\pi r$ long and h wide; thus the surface area of the can is

$$S = \pi r^2 + 2\pi r h \quad \hookrightarrow \quad S(r) = \pi r^2 + \frac{1000}{r}.$$

We have $S'(r) = 2\pi r - 1000/r^2$ for $r > 0$, and setting $S'(r) = 0$ yields $r = (500/\pi)^{1/3}$, in which case $h = \frac{500}{\pi}(500/\pi)^{-2/3} = (500/\pi)^{1/3}$. The can with minimum surface area therefore has height and radius $(500/\pi)^{1/3}$.

5 $f'(x) = -\sin x$, so that $f'(\pi/4) = -\frac{1}{\sqrt{2}}$. The linear approximation will be given by a line with slope $-\frac{1}{\sqrt{2}}$ and point $(\pi/4, f(\pi/4)) = \left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$, which is the linear function

$$L(x) = -\frac{1}{\sqrt{2}}x + \frac{\pi+4}{4\sqrt{2}}.$$

Now, $\cos 0.82 = f(0.82) \approx L(0.82) = 0.682640$.

6 Let $f(x) = 6x^5 + 13x + 1$, so equation becomes $f(x) = 0$. Since $f(-1) = -18 < 0$ and $f(0) = 1 > 0$, by the Intermediate Value Theorem there exists some $c \in (-1, 0)$ such that $f(c) = 0$, and this value would have to be a real root for the equation. Thus the equation must have at least one real root.

Suppose there exist two real roots $c_1 < c_2$ for the equation, so $f(c_1) = f(c_2) = 0$. Since the polynomial function f is everywhere continuous and differentiable, by Rolle's Theorem we conclude there must be some $r \in (c_1, c_2)$ for which $f'(r) = 0$. But this implies that $30r^4 + 13 = 0$, or $r^4 = -\frac{13}{30}$, so that r cannot be a real number, and thus cannot lie in the interval (c_1, c_2). Having arrived at a contradiction, we conclude that the equation cannot have two real roots, and therefore must have exactly one real root.

7a We apply L'Hôpital's Rule twice to resolve the 0/0 form:

$$\lim_{x\to 0} \frac{1-\cos 3x}{x^2} = \lim_{x\to 0} \frac{3\sin 3x}{2x} = \lim_{x\to 0} \frac{9\cos 3x}{2} = \frac{9\cos 0}{2} = \frac{9}{2}.$$

7b This is a $0 \cdot \infty$ form. We get a 0/0 form and apply L'Hôpital's Rule:

$$\lim_{x\to 0^+} \frac{\sin x}{\sqrt{\dfrac{x}{1-x}}} = \lim_{x\to 0^+} \frac{\cos x}{\dfrac{1}{2}\left(\dfrac{x}{1-x}\right)^{-1/2} \cdot \dfrac{1}{(1-x)^2}} = \lim_{x\to 0^+} 2\sqrt{x}(1-x)^{3/2}\cos x = 0.$$

8a $$\int (x^{3/4} + x^{5/2})\,dx = \frac{4}{7}x^{7/4} + \frac{2}{7}x^{7/2} + C.$$

8b $$\frac{1}{2}\int (3\cos\theta - \sin\theta)\,d\theta = \frac{3\sin\theta + \cos\theta}{2} + C.$$