Math 140 Exam #2 Key (Spring 2024)

1a $s'(\theta) = -\dfrac{12}{\theta^2} + \dfrac{12}{\theta^4} - \dfrac{4}{\theta^5}$.

1b Quotient Rule:

$$f'(x) = \frac{(2-x^3)(2x) - (-3x^2)(2+x^2)}{(2-x^3)^2} = \frac{x^4+6x^2+4x}{(2-x^3)^2}.$$

1c Product Rule:

$$y' = -\csc^3 x - \csc x \cot^2 x.$$

1d Quotient Rule:

$$h'(t) = \frac{(1-\sin t)(2\sec^2 t) - (2\tan t)(-\cos t)}{(1-\sin t)^2} = \frac{(1-\sin t)(2\sec^2 t) + 2\sin t}{(1-\sin t)^2}.$$

2 Seeking x for which $y'(x) = 4$, or $3x^2+1=4$, we get $x = \pm 1$. Since $y(1) = 2$, one point where slope is 4 is $(1,2)$, and the tangent line is $y = 4x-2$. Since $y(-1) = -2$, another point where slope is 4 is $(-1,-2)$, and the tangent line is $y = 4x+2$.

3a $y' = -3(4x-3x^5)^{-4}(4-15x^4)$.

3b $y' = \sec^2(\sqrt{x}\,)\cdot(\sqrt{x}\,)' = \dfrac{\sec^2(\sqrt{x}\,)}{2\sqrt{x}}$.

3c $h'(x) = 4\sin^3(\cos 7x)\cdot\cos(\cos 7x)\cdot(-\sin 7x)\cdot 7 = -28\cos(\cos 7x)\sin^3(\cos 7x)\sin 7x$.

4 We have

$$4x^3 - 2xy - x^2y' = 1 + 2y' \quad\hookrightarrow\quad y' = \frac{4x^3-2xy-1}{x^2+2}.$$

5 Implicit differentiation gives

$$y^{5/2} + \tfrac{5}{2}xy^{3/2}y' + \tfrac{3}{2}x^{1/2}y + x^{3/2}y' = 0,$$

and so

$$y' = -\frac{y^{5/2}+\frac{3}{2}x^{1/2}y}{x^{3/2}+\frac{5}{2}xy^{3/2}} = -\frac{2y^{5/2}+3x^{1/2}y}{2x^{3/2}+5xy^{3/2}}$$

At the point $(x,y) = (4,1)$ we have

$$y' = -\frac{2(1)^{5/2}+3(4)^{1/2}(1)}{2(4)^{3/2}+5(4)(1)^{3/2}} = -\frac{2+3(2)}{2(8)+5(4)} = -\frac{2}{9},$$

which is the slope of the tangent line. Equation of tangent line is therefore

$$y - 1 = -\tfrac{2}{9}(x-4) \quad \Rightarrow \quad 2x + 9y = 17.$$

6 Say one jet is a distance x from the intersection point, and the other is y away. Then $D = \sqrt{x^2+y^2}$ is the distance between the jets. We have $dx/dt = dy/dt = -700$ km/h, and so

$$\frac{dD}{dt} = \frac{1}{2}(x^2+y^2)^{-1/2}\left(2x\frac{dx}{dt} + 2y\frac{dy}{dt}\right) = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{\sqrt{x^2+y^2}} = -\frac{700(x+y)}{\sqrt{x^2+y^2}}$$

Finally, when $x = 14$ km and $y = 10$ km, we have

$$\left.\frac{dD}{dt}\right|_{(x,y)=(14,10)} = -\frac{700(14+10)}{\sqrt{14^2+10^2}} = -\frac{8400}{\sqrt{74}} \approx 976.5 \text{ km/h}.$$

7 From $V = \frac{4}{3}\pi r^3$ we have $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$, and with $dV/dt = 120\pi$ it follows that

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt} = \frac{30}{r^2}.$$

Now, when the radius is 5 cm it is increasing at a rate of

$$\left.\frac{dr}{dt}\right|_{r=5} = \frac{30}{5^2} = \frac{6}{5} \text{ cm/min}$$

Next, from $S = 4\pi r^2$ we have

$$\frac{dS}{dt} = 8\pi r\frac{dr}{dt} = 8\pi r \cdot \frac{30}{r^2} = \frac{240\pi}{r}.$$

When $r = 5$ cm the surface area is increasing at a rate of

$$\left.\frac{dS}{dt}\right|_{r=5} = \frac{240\pi}{5} = 48\pi \text{ cm}^2/\text{min}.$$