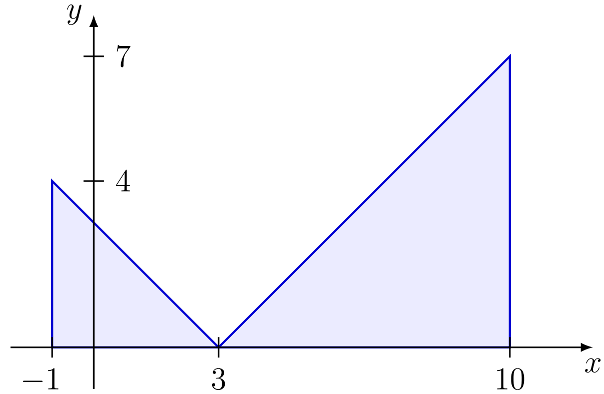
Math 140 Exam #4 Key (Spring 2022)

1a Find the areas of the triangles below: integral equals $\frac{65}{2}$.



1b We have

$$y = \sqrt{-x^2+6x-5} \quad \Rightarrow \quad y^2 = -x^2+6x-5 \quad \Rightarrow \quad (x^2-6x)+y^2 = -5 \quad \Rightarrow \quad (x-3)^2+y^2 = 4.$$

We are on the circle at $(3,0)$ with radius 2, but with $y \geq 0$ (square root can't be negative) and $1 \leq x \leq 5$ (the limits of the integral). The integral represents the area of the top half of the circle, and so equals 2π.

2a Expanding the product, we have

$$\int_0^4 (t^3 - 6t^2 + 8t)\,dt = \left[\frac{1}{4}t^4 - 2t^3 + 4t^2\right]_0^4 = 0.$$

2b $\displaystyle\int_1^8 x^{1/3}dx = \left[\frac{3}{4}x^{4/3}\right]_1^8 = \frac{45}{4}.$

2c $\displaystyle\int_{\pi/4}^{3\pi/4} \csc^2\theta\,d\theta = \big[-\cot\theta\big]_{\pi/4}^{3\pi/4} = \cot\frac{\pi}{4} - \cot\frac{3\pi}{4} = 1-(-1) = 2.$

3 By the Fundamental Theorem of Calculus and Chain Rule:

$$\frac{d}{dx}\int_{\cos x}^{9} \frac{6}{\sqrt{t^6+9}}\,dt = -\frac{d}{dx}\int_9^{\cos x} \frac{6}{\sqrt{t^6+9}}\,dt = -\frac{6}{\sqrt{\cos^6 x+9}}\cdot(\cos x)' = \frac{6\sin x}{\sqrt{\cos^6 x + 9}}.$$

4a Let $u = x^6 - 3x^2$, giving $du/dx = 6x^5 - 6x$ and thus $(x^5 - x)\,dx = \frac{1}{6}du$. The integral becomes

$$\int_0^{-2} \frac{1}{6}u^4\,du = \frac{1}{30}\big[u^5\big]_0^{-2} = -\frac{16}{15}.$$

4b Let $u = \sqrt{r}$, so $\frac{1}{\sqrt{r}} dr = 2du$ and the integral becomes

$$\int 2\cos u \, du = 2\sin u + C = 2\sin\sqrt{r} + C.$$

4c Let $u = x + 4$, so $du = dx$ and $x = u - 4$. The integral becomes

$$\int \frac{u-4}{\sqrt[3]{u}} du = \int \left(u^{2/3} - 4u^{-1/3}\right) du = \frac{3}{5}u^{5/3} - 6u^{2/3} + C = \frac{3}{5}(x+4)^{5/3} - 6(x+4)^{2/3} + C.$$

5 Area is

$$A = \int_0^2 \left[(3x - x^2) - x\right] dx + \int_2^3 \left[x - (3x - x^2)\right] dx = \frac{8}{3}.$$

6 Volume is

$$V = \int_0^3 A(x) \, dx = \int_0^3 \frac{1}{2}\pi\left(\frac{3-x}{2}\right)^2 dx = \frac{\pi}{8}\int_0^3 (x^2 - 6x + 9) \, dx = \frac{9\pi}{8}.$$

7 Disc Method here:

$$V = \int_2^4 \pi\left(\sqrt{25 - x^2}\right)^2 dx = \frac{94}{3}\pi.$$