Math 140 Exam #3 Key (Spring 2022)

1 We have $f'(x) = 1 + 2\sin x$, so $f'(x) = 0$ on $[-\pi, \pi]$ when $x = -\frac{\pi}{6}, -\frac{5\pi}{6}$. These are the critical points. We evaluate:

$$f(-\pi) = 2 - \pi, \quad f(\pi) = 2 + \pi, \quad f(-\tfrac{\pi}{6}) = -\sqrt{3} - \tfrac{\pi}{6}, \quad f(-\tfrac{5\pi}{6}) = \sqrt{3} - \tfrac{5\pi}{6}.$$

The absolute maximum value of f on $[-\pi, \pi]$ is $f(\pi) = 2 + \pi$, and the absolute minimum value is $f(-\frac{\pi}{6}) = -\sqrt{3} - \frac{\pi}{6}$.

2a Domain is $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$. The only intercept is $(0, 0)$.

2b Horizontal asymptote: $y = 1$. Vertical asymptotes: $x = \pm 2$.

2c Since

$$f'(x) = -\frac{8x}{(x^2 - 4)^2},$$

the only critical point of f is $x = 0$.

2d For x in the domain of f, we have $f'(x) > 0$ for $x < 0$, and $f'(x) < 0$ for $x > 0$. By the Monotonicity Test f is increasing on $(-\infty, -2) \cup (-2, 0)$, and decreasing on $(0, 2) \cup (2, \infty)$. By the First Derivative Test f has a local maximum at $(0, 0)$.

2e Here

$$f''(x) = \frac{24x^2 + 32}{(x^2 - 4)^3},$$

so $f''(x) < 0$ for $-2 < x < 2$, and $f''(x) > 0$ for $x < -2$ and $x > 2$. Therefore, by the Concavity Test, f is concave down on $(-2, 2)$, and concave up on $(-\infty, -2)$ and $(2, \infty)$. There are no inflection points.

3 A point on $y = -2x$ has the form $(x, -2x)$, and this point's distance from $(-20, 0)$ is

$$d(x) = \sqrt{(x + 20)^2 + (-2x)^2} = \sqrt{5x^2 + 40x + 400}.$$

We can minimize $d^2(x)$ a bit easier than $d(x)$ itself. Define

$$D(x) = d^2(x) = 5x^2 + 40x + 400.$$

Then $D'(x) = 0$ implies $10x + 40 = 0$, giving $x = -4$. The point on $y = -2x$ closest to $(-20, 0)$ is therefore $(-4, 8)$. Distance between these points is $\sqrt{16^2 + 8^2} = 8\sqrt{5}$.

4 Say there are two fences of length x, and four fences of length y (which includes the two interior fences). We have $2x + 4y = 400$, or $x = 200 - 2y$. The area of the enclosed field is $A(y) = xy = -2y^2 + 200y$. Now, $A'(y) = -4y + 200$, so $A'(y) = 0$ implies $y = 50$. This corresponds to a maximum value for $A(y)$. With $y = 50$ we have $x = 100$, so the dimensions of the rectangle with maximum area is 100 ft $\times$ 50 ft, and the area is 5000 ft^2.

5 Let $f(x) = \sqrt[3]{x}$, so $f'(x) = \frac{1}{3}x^{-2/3}$. Since $\sqrt[3]{8} = 2$, and 8 is near 7, we find a linearization for f at $x = 8$. This is

$$L(x) = f'(8)(x-8) + f(8) = \frac{x}{12} + \frac{4}{3}.$$

Now,

$$\sqrt[3]{7} = f(7) \approx L(7) = \frac{7}{12} + \frac{4}{3} = \frac{23}{12} = 1.91\overline{6}.$$

6 Let $f(x) = x^5 + 10x + 3$, so equation becomes $f(x) = 0$. Since $f(-1) = -8 < 0$ and $f(0) = 3 > 0$, by the Intermediate Value Theorem there exists some $c \in (-1, 0)$ such that $f(c) = 0$, and this value would have to be a real root for the equation. That is, the equation is sure to have at least one real root.

Suppose there exist two real roots $c_1 < c_2$ for the equation, so $f(c_1) = f(c_2) = 0$. Since the polynomial function f is everywhere continuous and differentiable, by Rolle's Theorem we conclude there must be some $r \in (c_1, c_2)$ for which $f'(r) = 0$. But this implies that $5r^4 + 10 = 0$, or $r^4 = -2$, so that r cannot be a real number, and thus it cannot lie in the interval (c_1, c_2). Having arrived at a contradiction, we conclude that the equation cannot have two real roots, and therefore must have exactly one real root.

7a We have

$$\lim_{x\to 0} \frac{\sin ax}{\sin bx} \overset{\text{LR}}{=} \lim_{x\to 0} \frac{a\cos ax}{b\cos bx} = \frac{a\cos 0}{b\cos 0} = \frac{a}{b}.$$

7b Get a common denominator and use L'Hôpital's Rule twice:

$$\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{\sin x}\right) = \lim_{x\to 0} \frac{\sin x - x}{x\sin x} \overset{\text{LR}}{=} \lim_{x\to 0} \frac{\cos x - 1}{x\cos x + \sin x}$$

$$\overset{\text{LR}}{=} \lim_{x\to 0} \frac{\sin x}{x\sin x - 2\cos x} = \frac{0}{0-2} = 0.$$

8a $$\int\left(\frac{5}{t^2} + 4t^2\right)dt = \int(5t^{-2} + 4t^2)dt = -\frac{5}{t} + \frac{4}{3}t^3 + C.$$

8b $$\int(\cos 2x - \csc^2 8x)\,dx = \frac{1}{2}\sin 2x + \frac{1}{8}\cot 8x + C.$$