Math 140 Exam #3 Key (Spring 2021)

1 We have $f'(x) = 4x^3 - 12x^2 + 8x = 4x(x-1)(x-2)$, so $f'(x) = 0$ when $x = 0, 1, 2$. These are the critical points. We evaluate:

$$f(-1) = 9, \quad f(0) = 0, \quad f(1) = 1, \quad f(2) = 0, \quad f(3) = 9.$$

The absolute minimum value of f on $[-1, 3]$ is $f(0) = f(2) = 0$, and the absolute maximum value is $f(-1) = f(3) = 9$.

2a Domain is $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$. The only intercept is $(0, 0)$.

2b Horizontal asymptote: $y = 1$. Vertical asymptotes: $x = \pm 2$.

2c Since

$$f'(x) = -\frac{8x}{(x^2-4)^2},$$

the only critical point of f is $x = 0$.

2d For x in the domain of f, we have $f'(x) > 0$ for $x < 0$, and $f'(x) < 0$ for $x > 0$. By the Monotonicity Test f is increasing on $(-\infty, -2) \cup (-2, 0)$, and decreasing on $(0, 2) \cup (2, \infty)$. By the First Derivative Test f has a local maximum at $(0, 0)$.

2e Here

$$f''(x) = \frac{24x^2 + 32}{(x^2-4)^3},$$

so $f''(x) < 0$ for $-2 < x < 2$, and $f''(x) > 0$ for $x < -2$ and $x > 2$. Therefore, by the Concavity Test, f is concave down on $(-2, 2)$, and concave up on $(-\infty, -2)$ and $(2, \infty)$. There are no inflection points.

3 A point on $y = -2x$ has the form $(x, -2x)$, and this point's distance from $(-20, 0)$ is

$$d(x) = \sqrt{(x+20)^2 + (-2x)^2} = \sqrt{5x^2 + 40x + 400}.$$

We can minimize $d^2(x)$ a bit easier than $d(x)$ itself. Define

$$D(x) = d^2(x) = 5x^2 + 40x + 400.$$

Then $D'(x) = 0$ implies $10x + 40 = 0$, giving $x = -4$. The point on $y = -2x$ closest to $(-20, 0)$ is therefore $(-4, 8)$. Distance between these points is $\sqrt{16^2 + 8^2} = 8\sqrt{5}$.

4 Let x be as in the figure below (i.e. x is the distance between the nearest point on the shore to the island and the point where the cable will meet the shore). Cost function is:

$$C(x) = \left(\frac{\$2400}{\text{km}}\right)\left(\sqrt{x^2 + 3.5^2}\text{ km}\right) + \left(\frac{\$1200}{\text{km}}\right)(8 - x\text{ km}) = 2400\sqrt{x^2 + \frac{49}{4}} + 1200(8 - x).$$

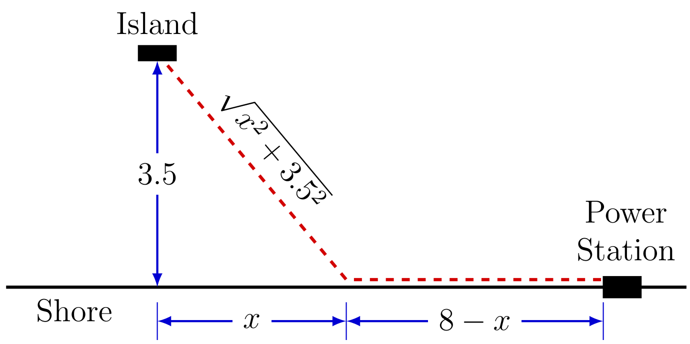
We take the derivative:

$$C'(x) = \frac{2400x}{\sqrt{x^2 + 49/4}} - 1200.$$

Note that there is no x value for which $C'(x)$ does not exist. On the other hand,

$$C'(x) = 0 \;\Rightarrow\; \frac{2x}{\sqrt{x^2+49/4}} - 1 = 0 \;\Rightarrow\; 2x = \sqrt{x^2 + \frac{49}{4}} \;\Rightarrow\; 4x^2 = x^2 + \frac{49}{4}$$
$$\Rightarrow\; x^2 = \frac{49}{12} \;\Rightarrow\; x = \sqrt{\frac{49}{12}} = \frac{7}{2\sqrt{3}} = \frac{7\sqrt{3}}{6}.$$

Thus, if the cable meets the shore at a point about $8 - 7\sqrt{3}/6$ km to the left of the power station, cost will be minimized.



5 We have $g'(t) = 1/\sqrt{2t+9}$. The linear approximation in question is the line through $(-4, g(-4)) = (-4, 1)$ with slope $g'(-4) = 1$, which has equation $y = t + 5$, and so the function $L(t) = t + 5$ is the linear approximation (or "linearization") of $g(t)$ at $t = -4$.

6 Let $f(x) = x^5 + 10x + 3$, so equation becomes $f(x) = 0$. Since $f(-1) = -8 < 0$ and $f(0) = 3 > 0$, by the Intermediate Value Theorem there exists some $c \in (-1, 0)$ such that $f(c) = 0$, and this value would have to be a real root for the equation. That is, the equation is sure to have at least one real root.

Suppose there exist two real roots $c_1 < c_2$ for the equation, so $f(c_1) = f(c_2) = 0$. Since the polynomial function f is everywhere continuous and differentiable, by Rolle's Theorem we conclude there must be some $r \in (c_1, c_2)$ for which $f'(r) = 0$. But this implies that $5r^4 + 10 = 0$, or $r^4 = -2$, so that r cannot be a real number, and thus it cannot lie in the interval (c_1, c_2). Having arrived at a contradiction, we conclude that the equation cannot have two real roots, and therefore must have exactly one real root.

7a We have

$$\lim_{x\to 0} \frac{\sin ax}{\sin bx} \overset{\text{LR}}{=} \lim_{x\to 0} \frac{a\cos ax}{b\cos bx} = \frac{a\cos 0}{b\cos 0} = \frac{a}{b}.$$

7b Get a common denominator and use L'Hôpital's Rule twice:

$$\begin{aligned}\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right) &= \lim_{x\to 0}\frac{\sin x - x}{x\sin x} \overset{\text{LR}}{=} \lim_{x\to 0}\frac{\cos x - 1}{x\cos x + \sin x} \\ &\overset{\text{LR}}{=} \lim_{x\to 0}\frac{\sin x}{x\sin x - 2\cos x} = \frac{0}{0-2} = 0.\end{aligned}$$

8a $\displaystyle\int\left(\frac{5}{t^2}+4t^2\right)dt = \int(5t^{-2}+4t^2)dt = -\frac{5}{t}+\frac{4}{3}t^3+C.$

8b $\displaystyle\int(\sin 2r - \sec^2 9r)\,dr = -\frac{1}{2}\cos 2r - \frac{1}{9}\tan 9r + C.$